Network Algorithms

Distributed Counting

Distributed Counting

Counter is a shared variable, supporting atomic Test-and-Increment. The requesting processor Gets the value and the counter is then incremented.

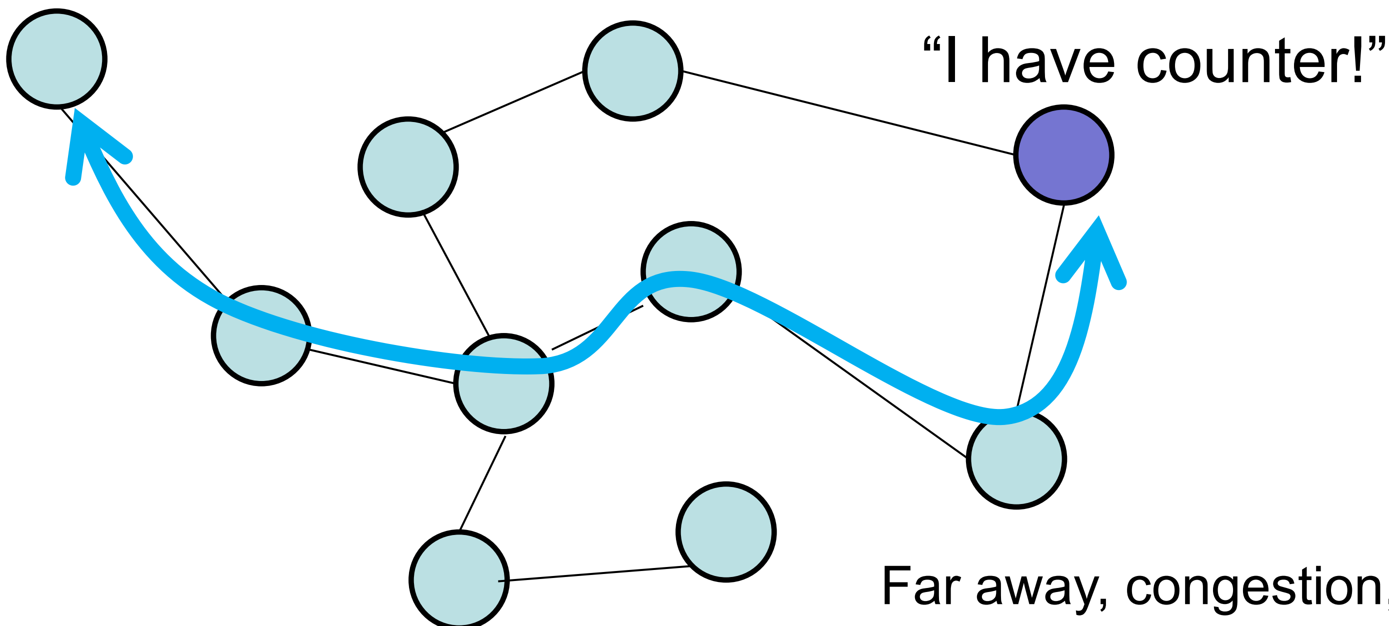
Applications? E.g., load-balancing!
Send to server i (mod n)…
Or: renaming identifiers to {1,…,n}

How to implement?

Simple Solution

Simple Algo

Send to some node v, v replies and increments.



Far away, congestion, …:
Not very scalable ☹
Can I count in a decentralized way??

A crazy idea?

Batcher networks permute values: use it for balancing the counting?!

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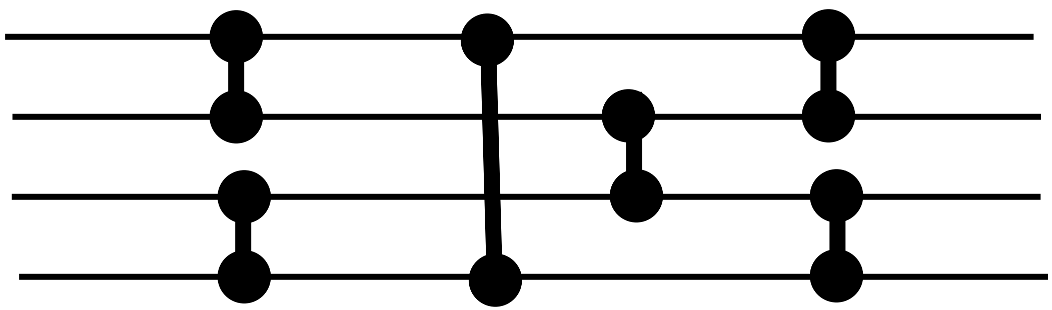
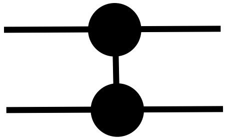


Idea 1: replace comparators with perfect balancers!

Balancer (B)

Values at “in” are sent to upper and lower “out” in turns:

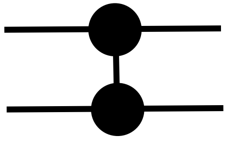
Simply a “toggler”:



Balancer Example

Assume this time of arrival:

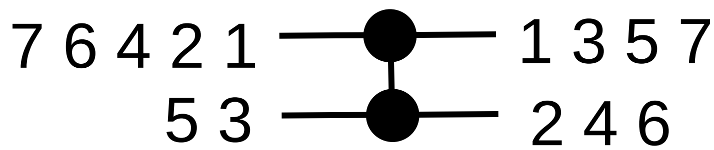
7 6 4 2 1
5 3



Order at output?

Balancer Example

Assume this time of arrival:



The top wire has one more than the lower wire!
Coincidence?

Balancer Example

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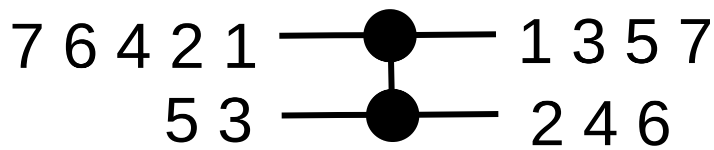
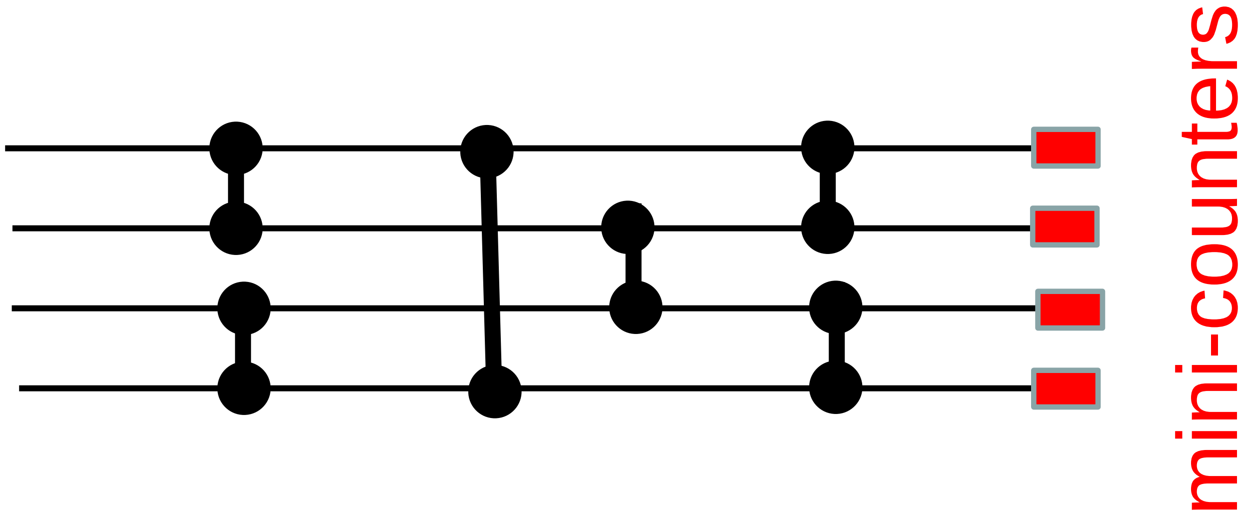


The top wire has one more than the lower wire!
Coincidence?

No, will always be almost perfectly balanced!
Since first goes to top: top may have 1 more in case of an odd number of requests…

A crazy idea?

Batcher networks permute values: use it for balancing the counting?!



Idea 2: add counters at end!

Mini-Counters: distributed!

A crazy idea?

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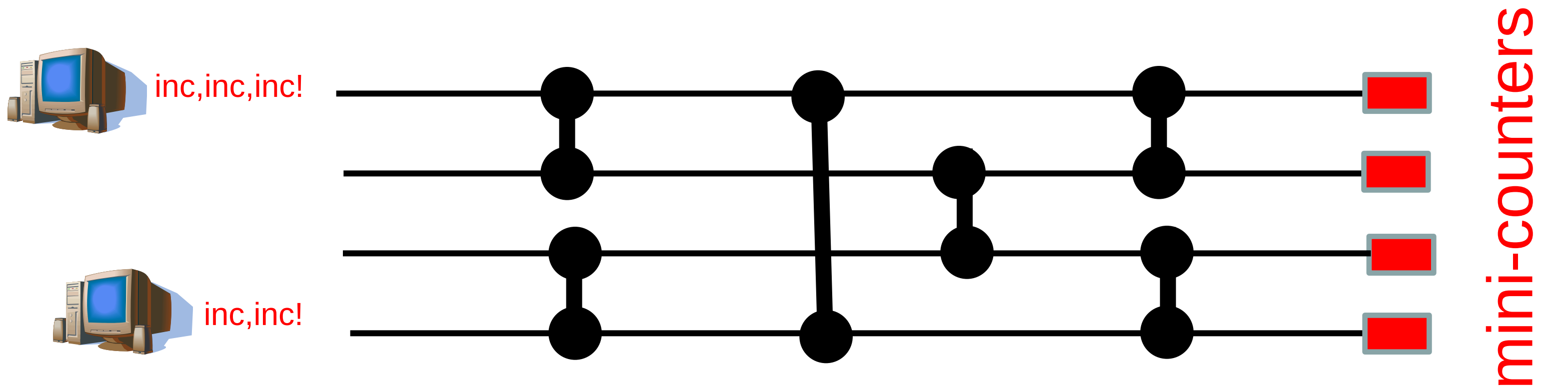

Idea 3: to count, just send
a request through network!
(Routed by balancers…)

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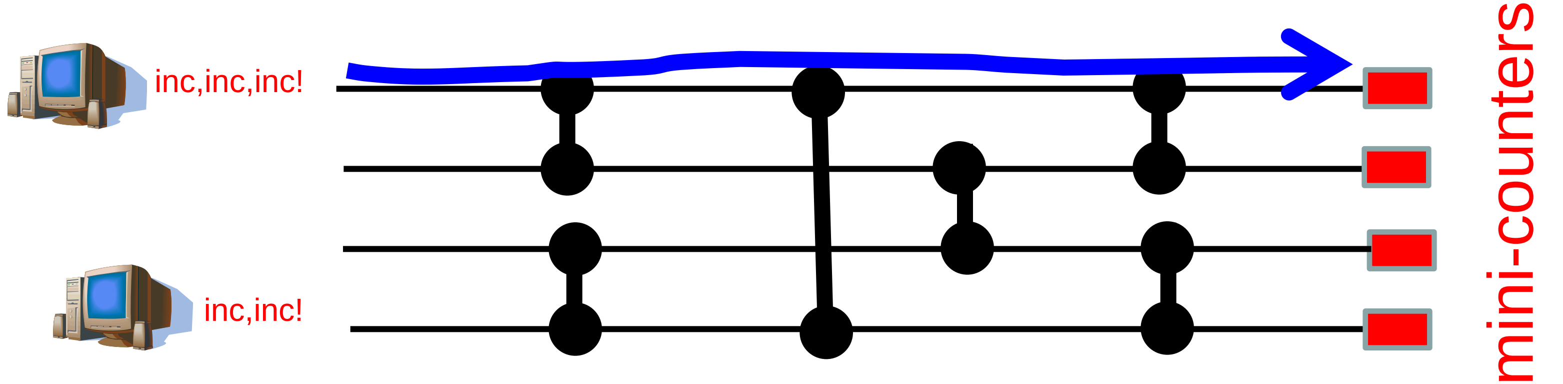

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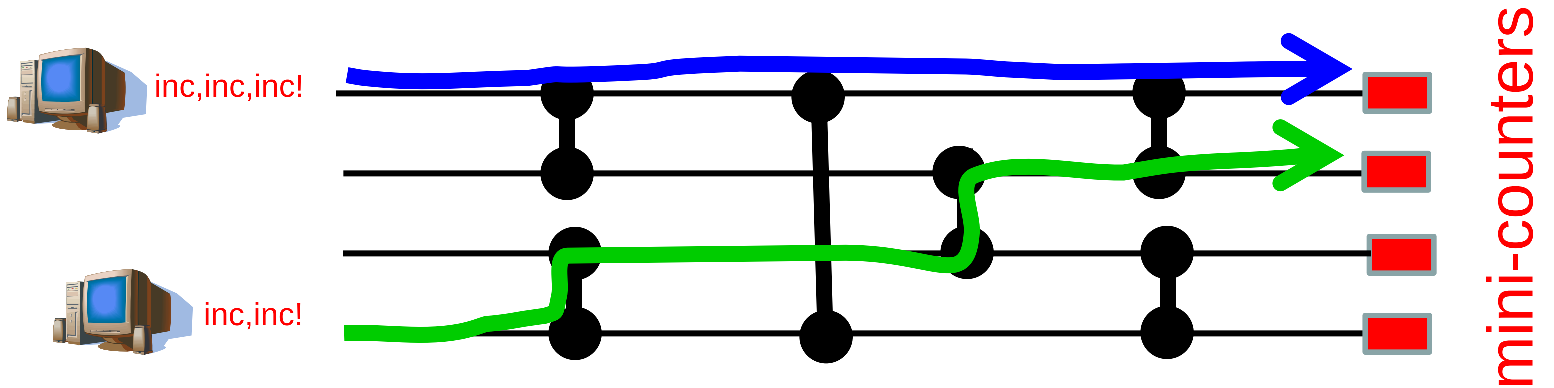

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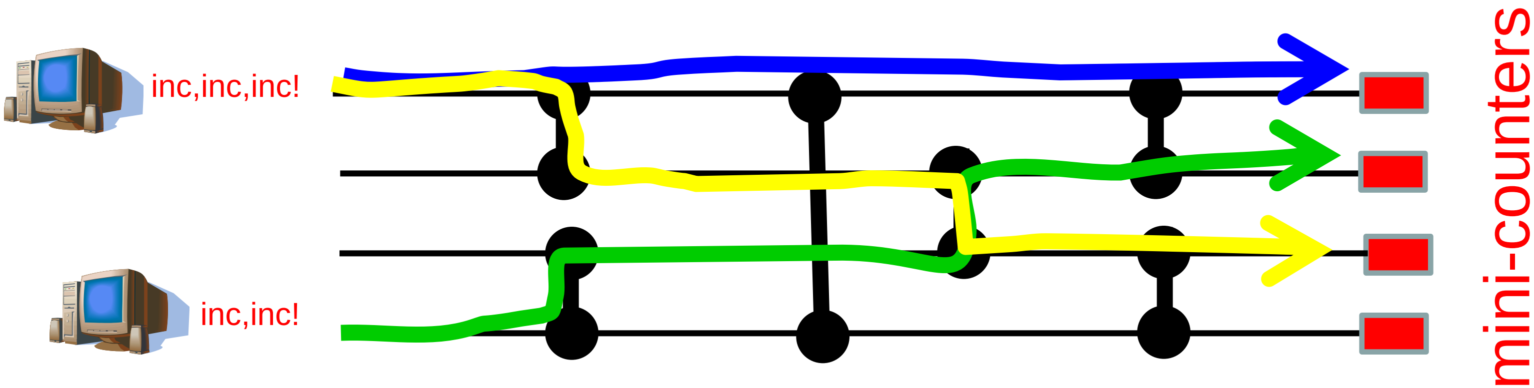

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a request through network!
(Routed by balancers…)

The BCN: Overview

Bitonic Counting Network (BCN)

1. Take Batcher and replace comparators with balancers
2. When a node wants to count: sends message to an arbitrary input wire
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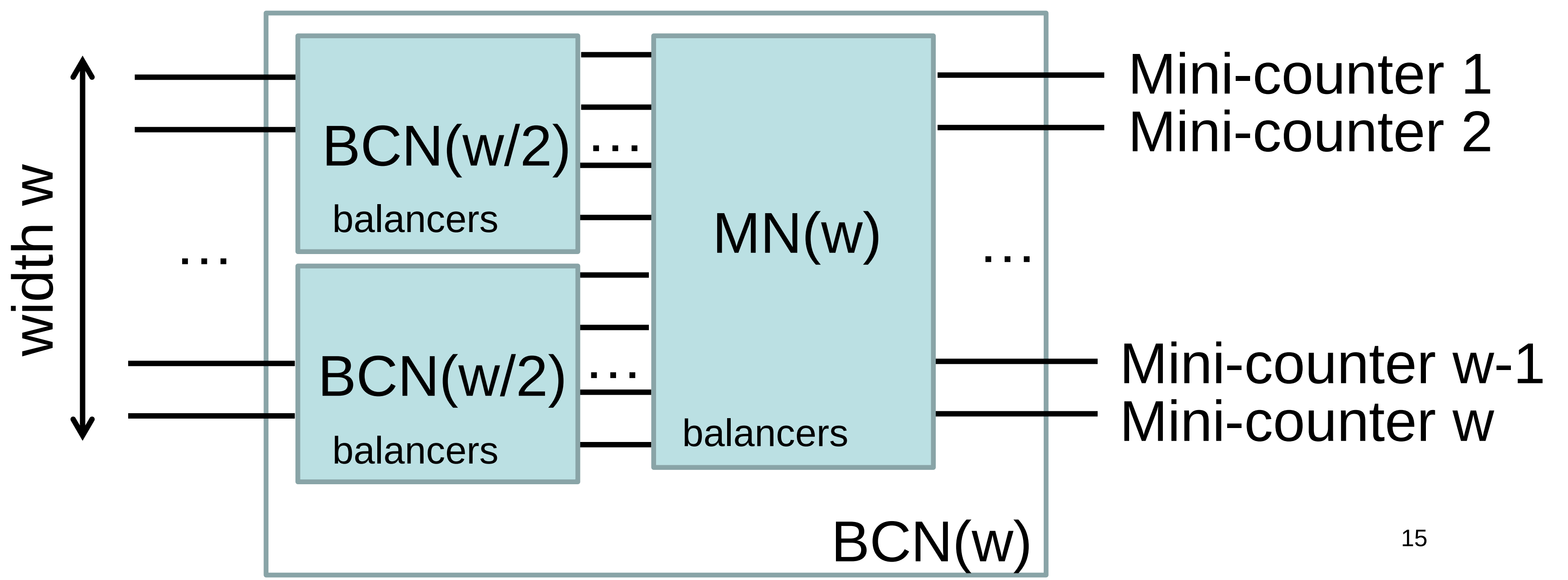
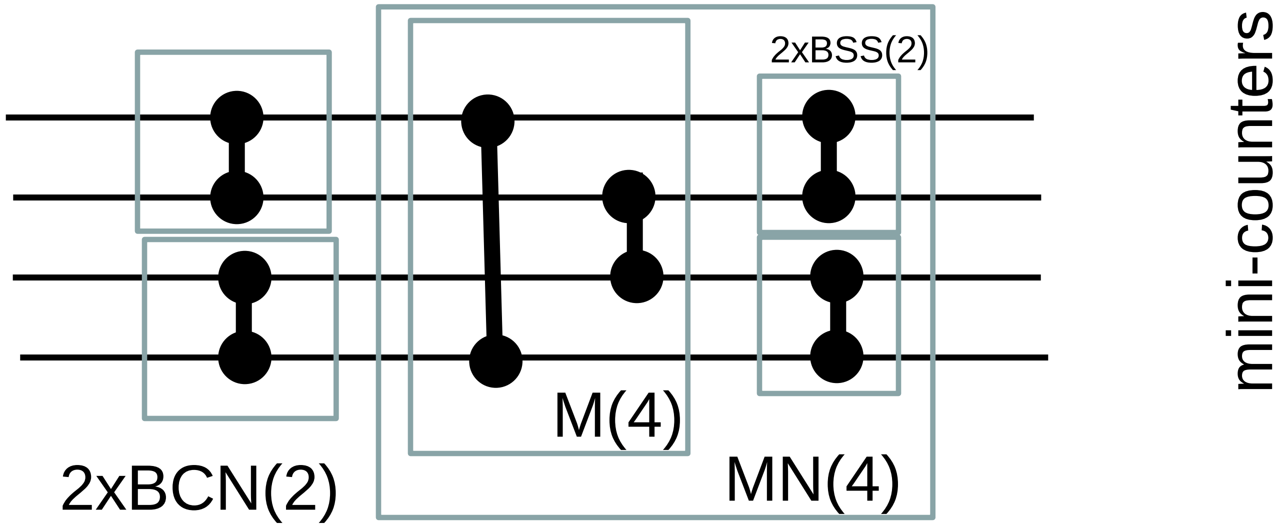


Example

Bitonic Counter Network BCN(4):



Example

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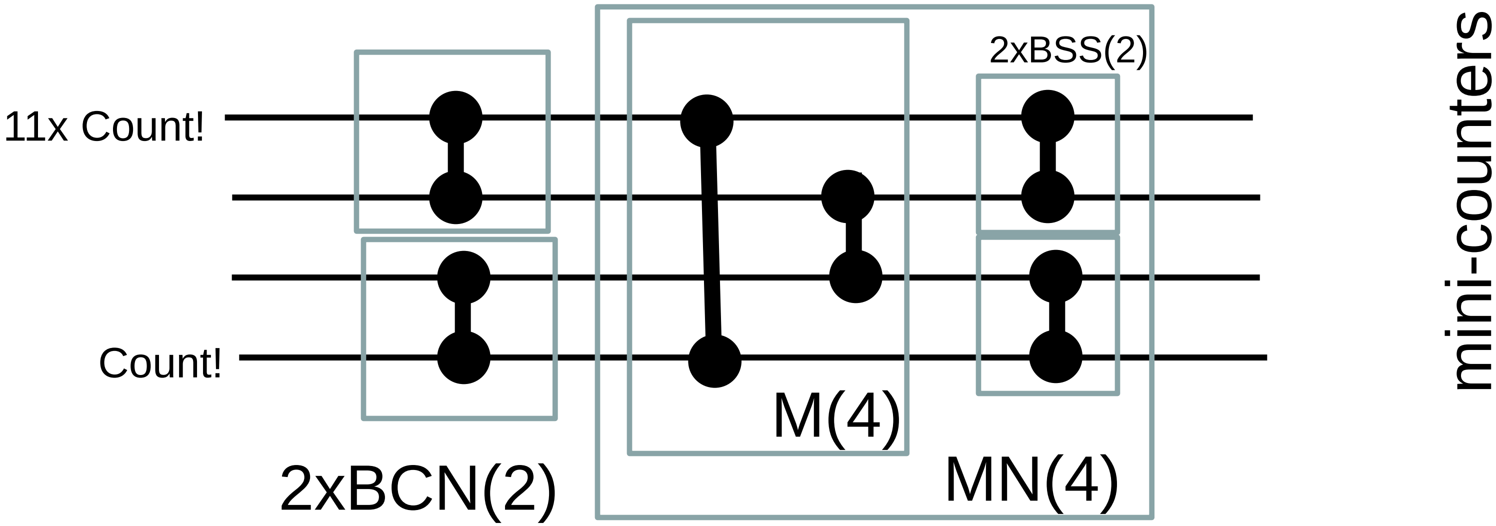
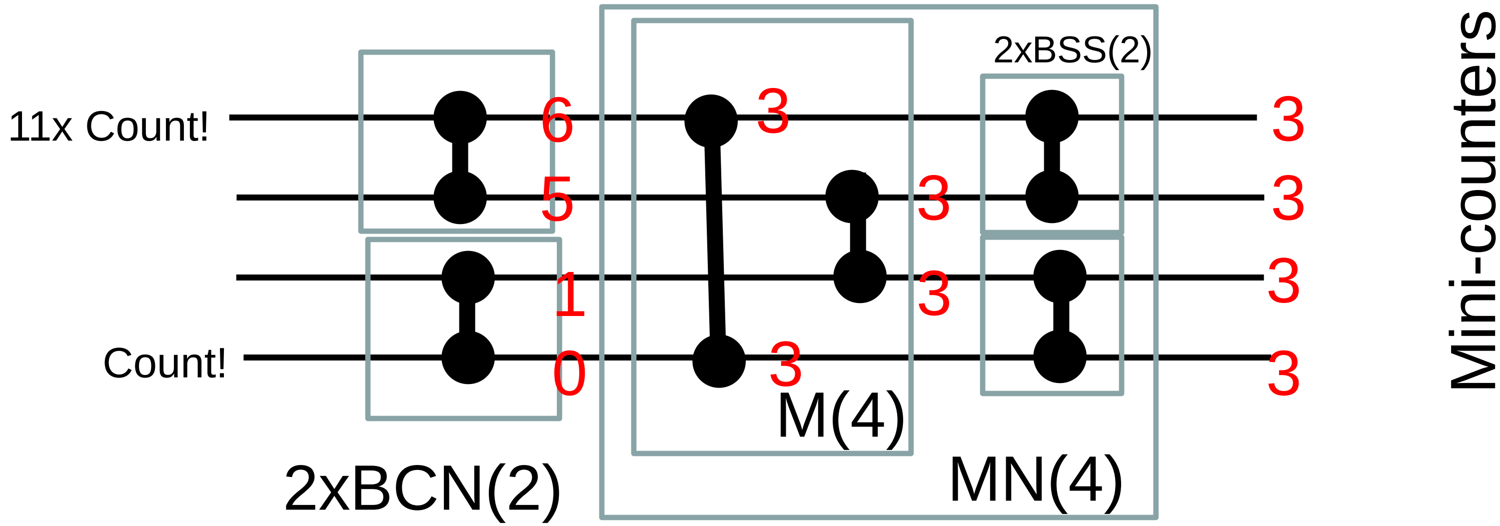


Example

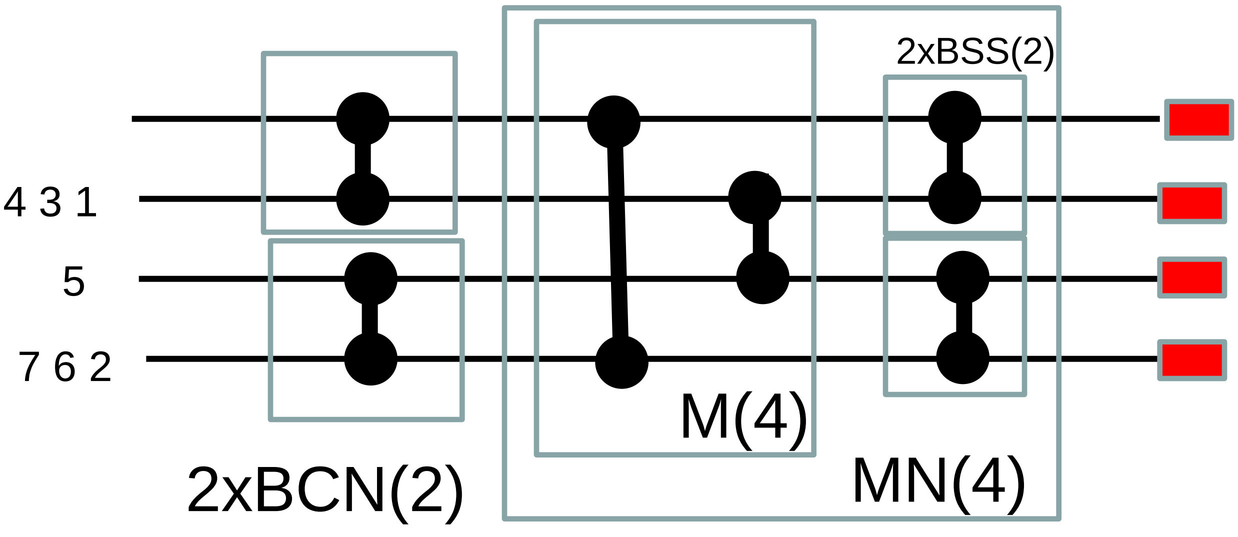
Bitonic Counter Network BCN(4):



So depth / increment time? $\log^2 n$, as for sorting.

Example With Requests

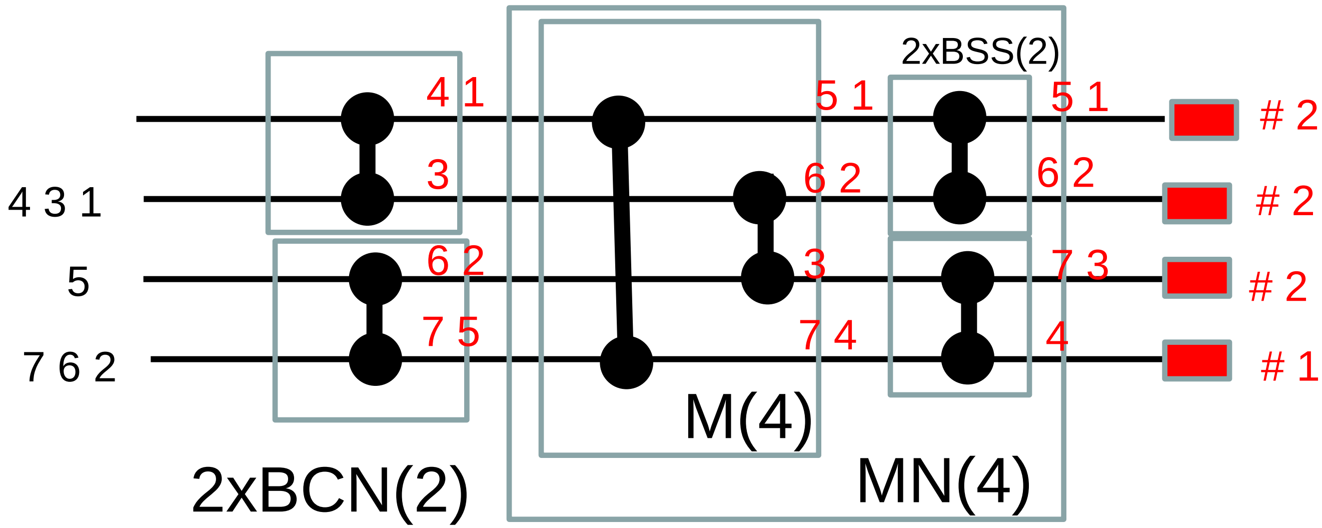
General pattern:



Request propagation?

Example With Requests

General pattern:



Frequency distribution (2,2,2,1): called Step Property!
Formula for requests arriving at i-th counter from top?

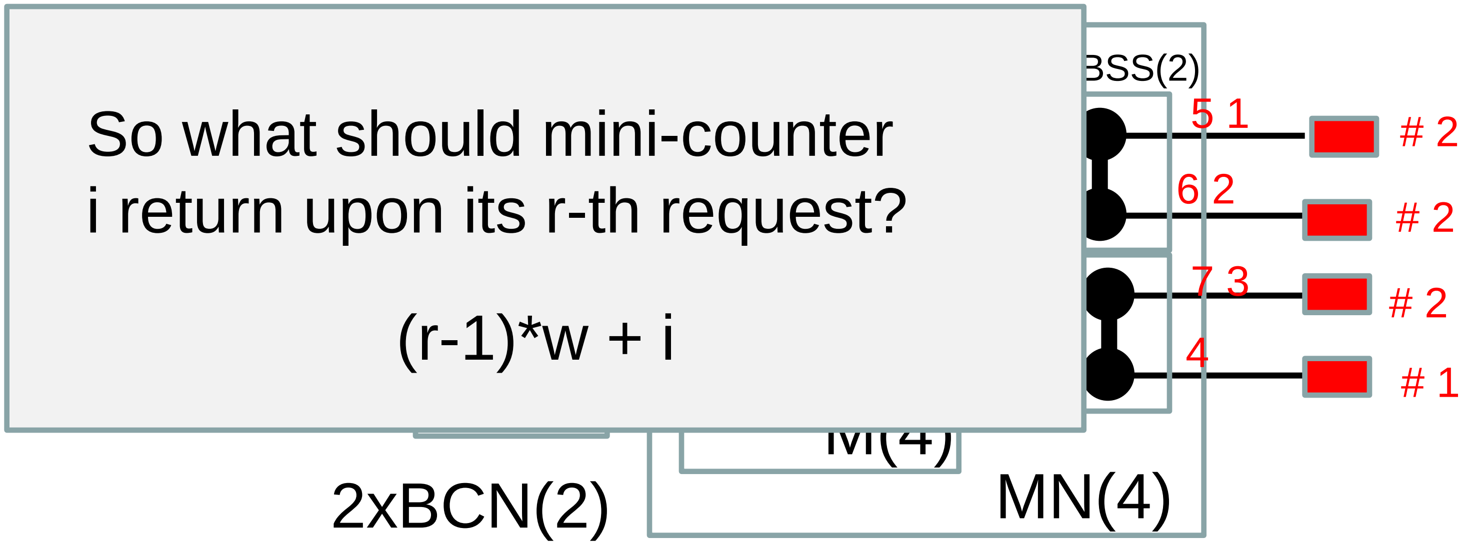
Token t mod w = i, every i-th request!
Perfectly load balanced and consistent ☺

Example With Requests

General pattern:

So what should mini-counter i return upon its r-th request?

(r-1)*w + i



Frequency distribution (2,2,2,1): called Step Property!
Formula for requests arriving at i-th counter from top?

Token t mod w = i, every i-th request!
Perfectly load balanced and consistent ☺

Analysis

Correctness

In a quiescent state, the w output wires of a bitonic counting network of width w have the step property.

Step Property

A sequence $y_0,\ldots,y_{w-1}$ has the step property iff:
For any i<j: $0 \leq y_i - y_j \leq 1$

Quiescent State

No message in transit.

Example:
8,8,8,8,7,7,7,7,7 ☺
Idea: if output wires have this property, exactly the values 1, …, r will be assigned by the mini-counters!

8
8
8
7

Note: Step property implies that values differ by at most 1, and large values come first!

Balancer Properties

Balancer Properties

Let x_i denote the number of messages consumed on input i, where i={0,1}. Denote y_i denote the number of messages on output wire i. Then:

(A1) $x_0+x_1 \geq y_0+y_1$

(A2) In a quiescent state: $x_0+x_1 = y_0+y_1$

(A3) The number of messages sent to the upper output wire is at most one higher than the number on the lower: $y_0 = \text{ceil}((y_0+y_1)/2)$, $y_1 = \text{floor}((y_0+y_1)/2)$

We assume that message is sent to upper first.

Some Facts (1)

Step Property Properties

If a sequence $y_0, y_{1,\ldots}, y_{w-1}$ has step property,
(B1) All subsequences have step property too
(B2) And even and odd subsequences satisfy:

$$\sum_{i=0}^{w/2-1} y_{2i} = \left\lceil \frac{1}{2} \sum_{i=0}^{w-1} y_i \right\rceil \; and \; \sum_{i=0}^{w/2-1} y_{2i+1} = \left\lfloor \frac{1}{2} \sum_{i=0}^{w-1} y_i \right\rfloor$$

Example: 7,7,7,6,6,6,6,6 = 51

25= floor(51/2)

26= ceil(51/2)

More Step Properties

Given two sequences $x_0,x_{1,\ldots},x_{w-1}$ and $y_0,y_{1,\ldots},y_{w-1}$ with the step property,

(C1) If $\sum x_i = \sum y_i$, then $x_i=y_i$ for any i

(C2) If $\sum x_i = \sum y_i+1$, then there exists a unique j such that $x_j=y_j+1$, and $x_i=y_i$ for any other i.

Example: x = 7,7,7,7,6,6,6,6 = 52
y = 7,7,7,6,6,6,6,6 = 51

Correctness

The step property is maintained inductively!

Proof. Also by induction over balancers, s. lecture notes. ∎

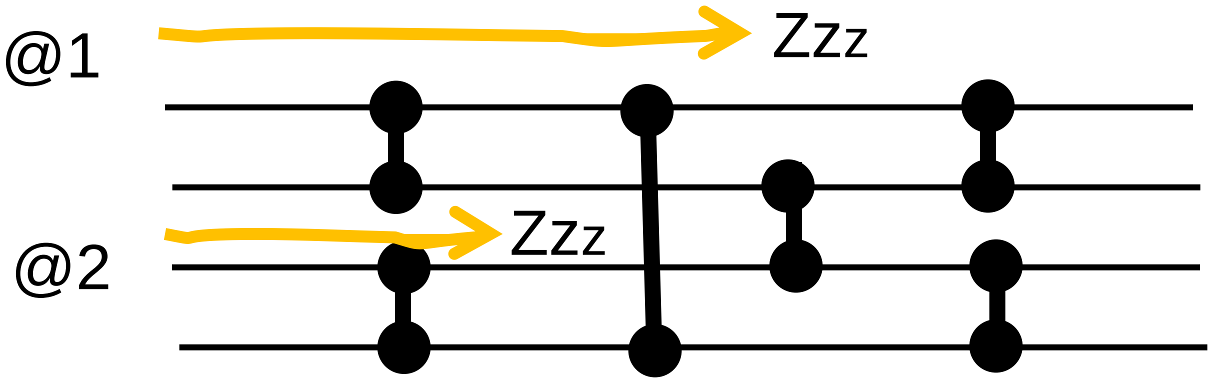
However, only correct in "quiescence state". But: Counters will always be unique and there are no gaps (gap means request still in transit!), just sometimes a larger counter is issued before a smaller counter!

Linearizability

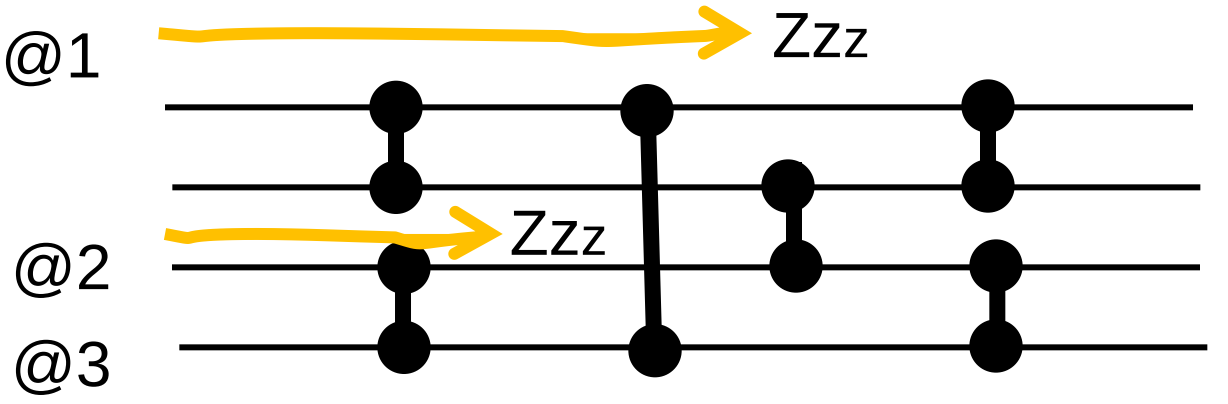
A system is linearizable if the order in which values are assigned reflects the real-time order in which they were requested: if there is a pair of operations o1, o2, where operation o1 terminates before operation o2 starts, and the logical order is “o2 before o1”, then a distributed system is not linearizable.

Is Bitonic Counting Network linearizable?

Not linearizable!

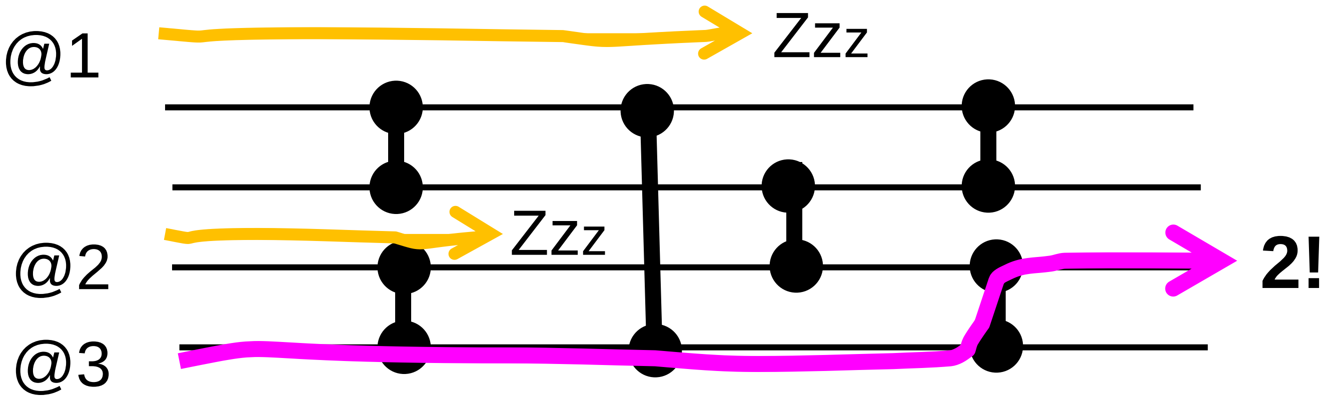


Not linearizable!



What happens?

Not linearizable!



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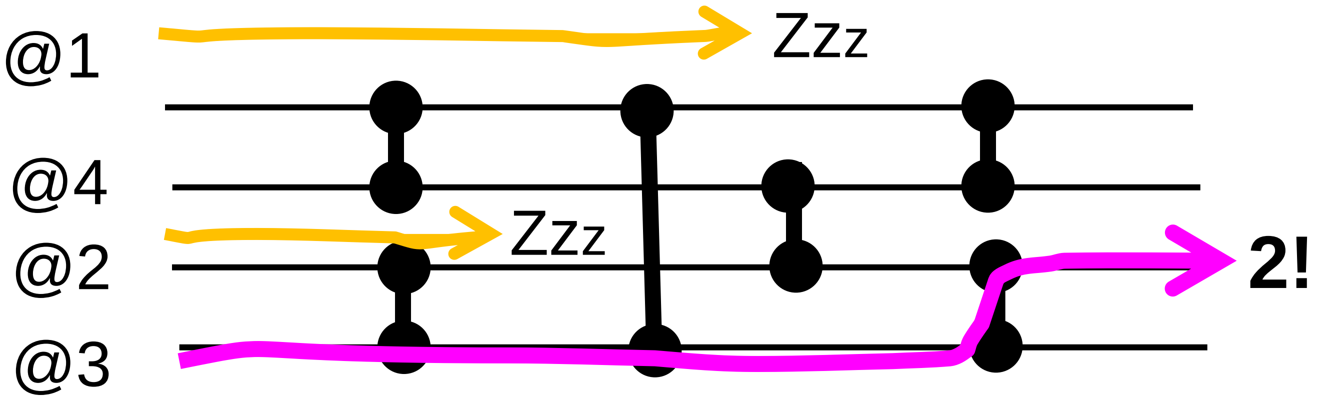
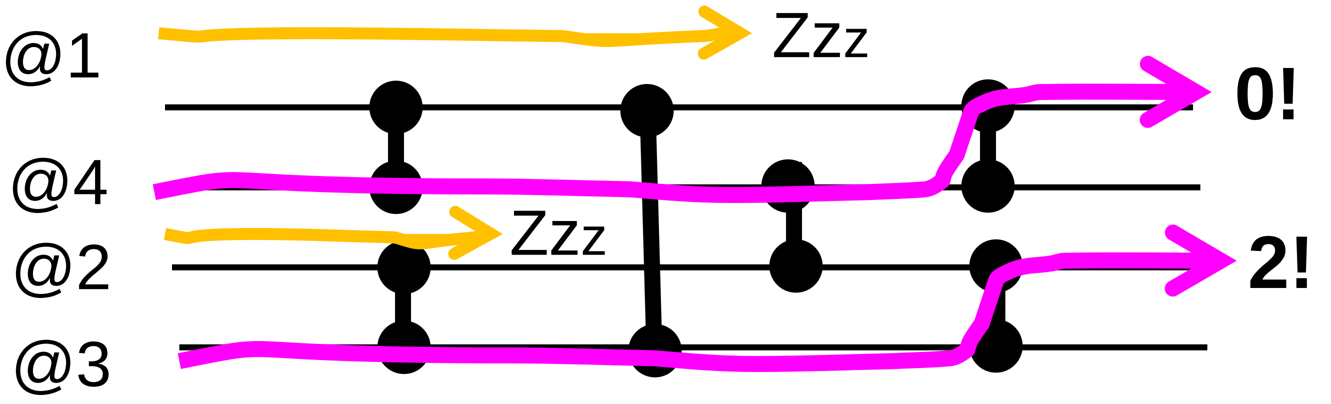
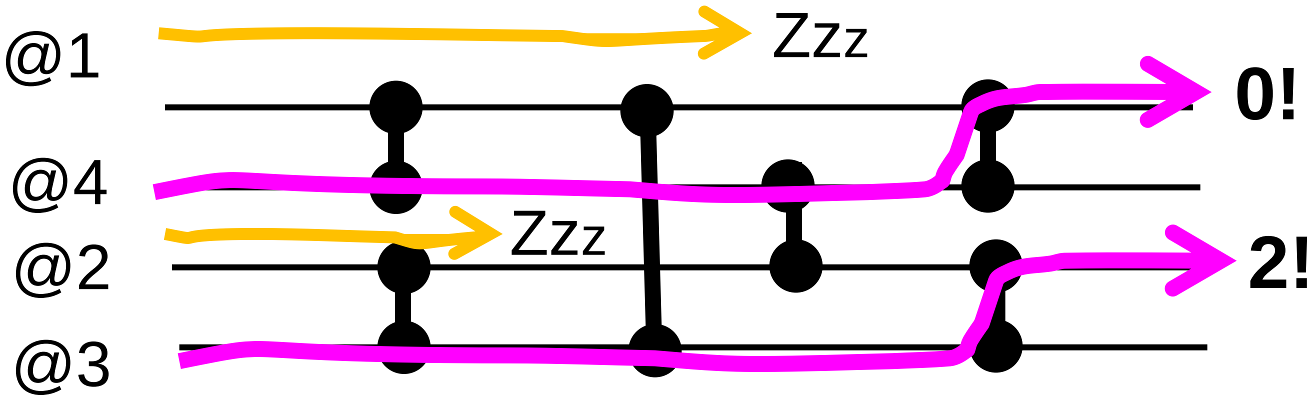


What happens now?

Not linearizable!



Not linearizable!



Request started and finished earlier, but got higher number...

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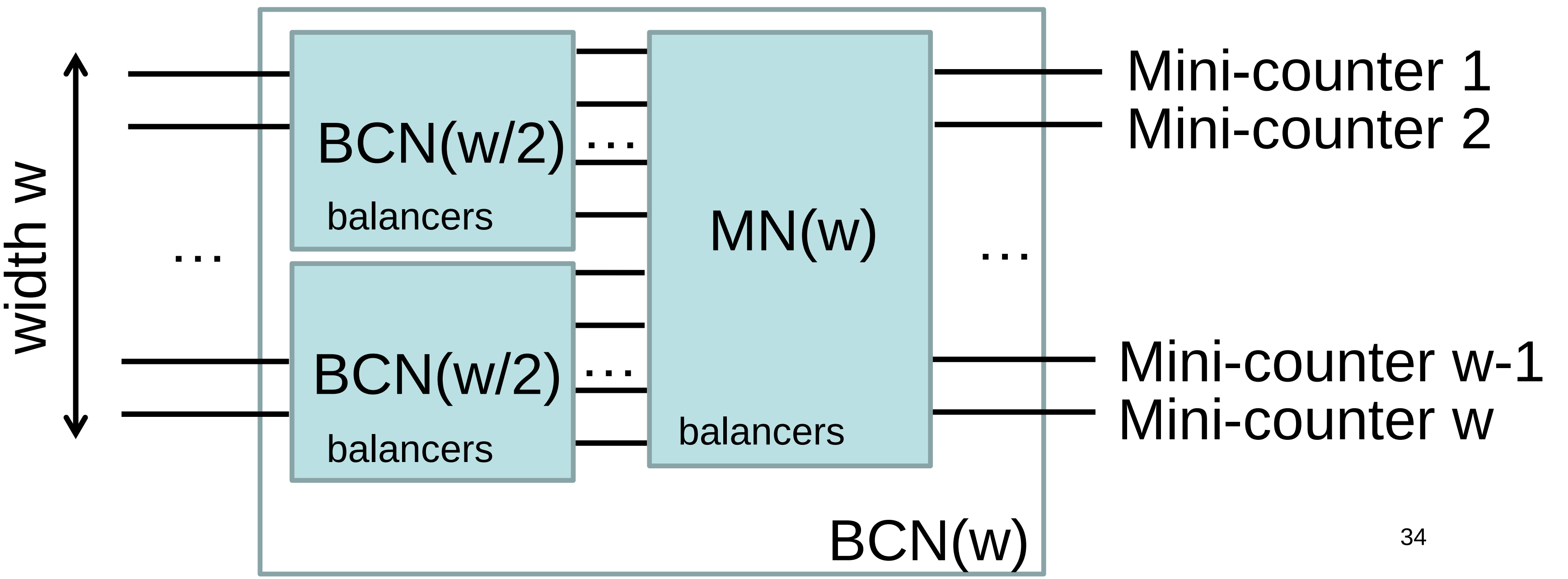
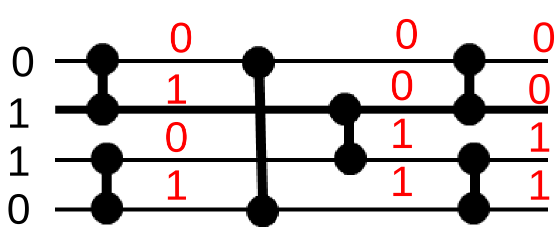


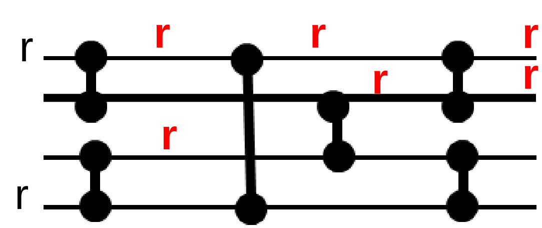
Difference Between Sorting and Counting?

Counting vs Sorting

Any counting network C is also a sorting network I(C), when replacing balancers with comparators.

Proof. Assume: want to sort binary vector {0,1}* in I(C).
Make a “request” on each wire of C if there is a “0” at corresponding wire of I(C).
Since C is a counting network, all requests end up at upper wires.
A comparator of I(C) will receive a “0” on its upper wire iff the corresponding balancer in C receives a request on upper wire (correspondence between requests and “0”); similarly for lower wire.
By induction: 0s always exit upper wires (like requests). ∎

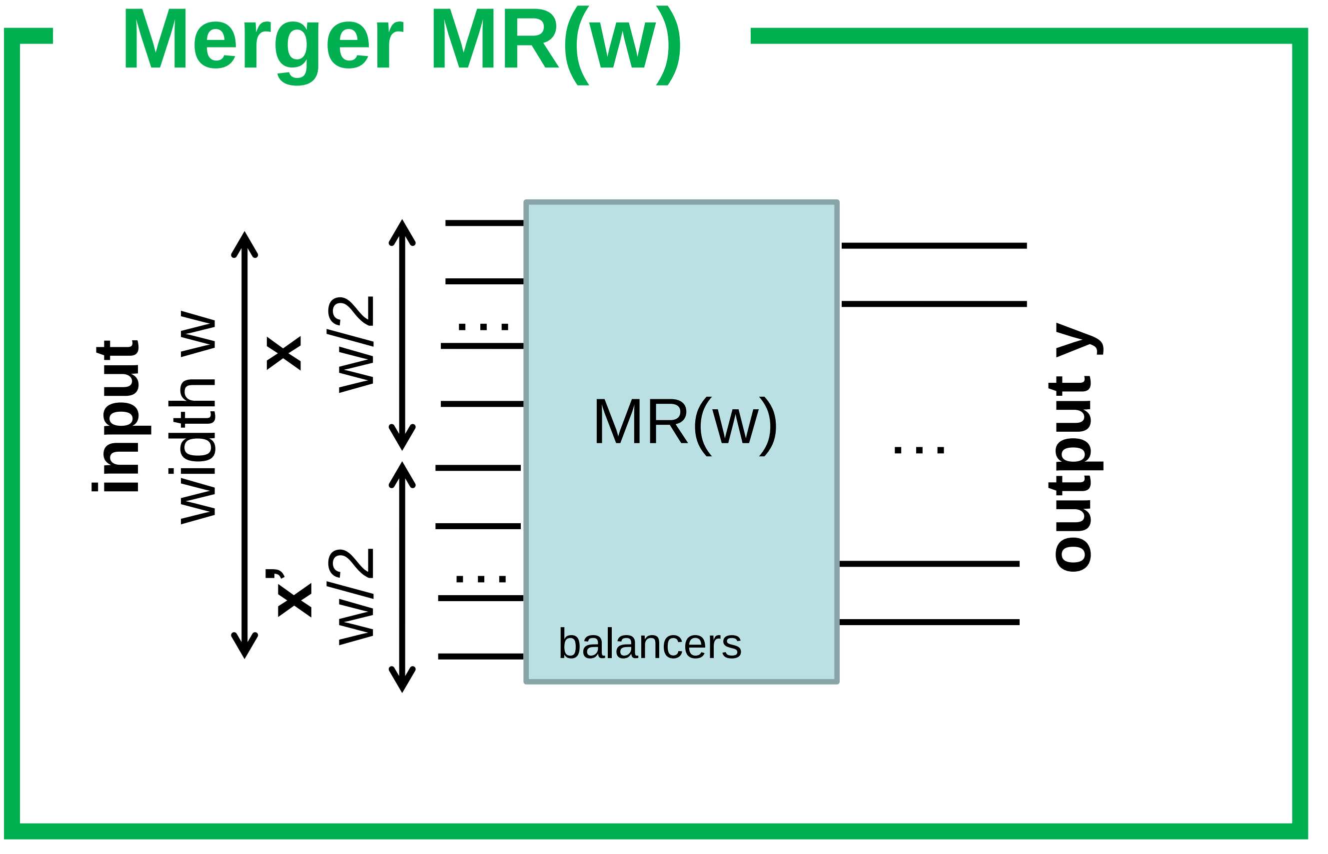




The other direction is not true (already Odd/Even network)…

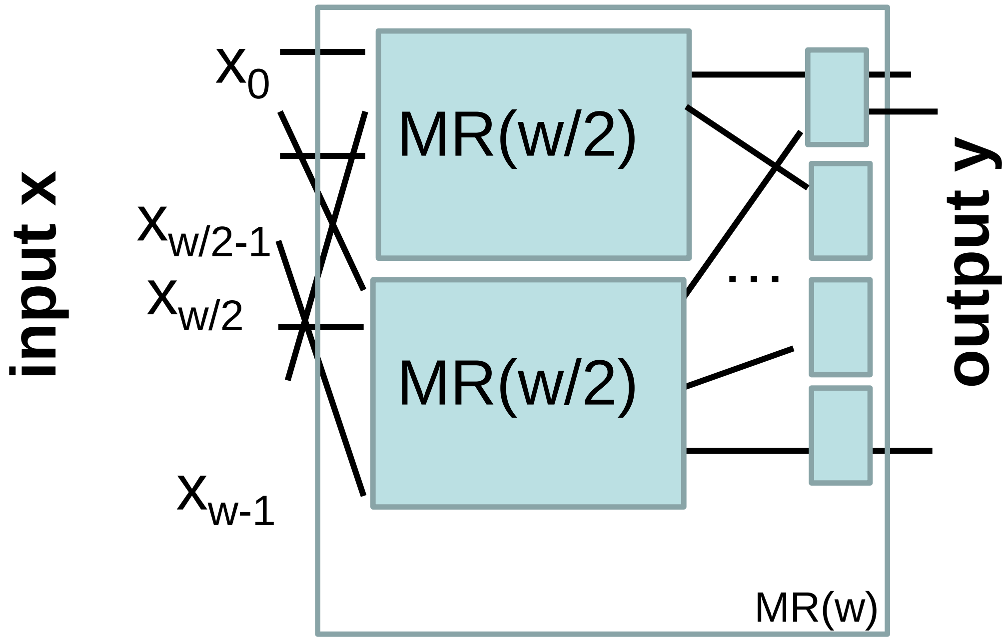
End of Lecture

Backup: Proof



Backup: Proof

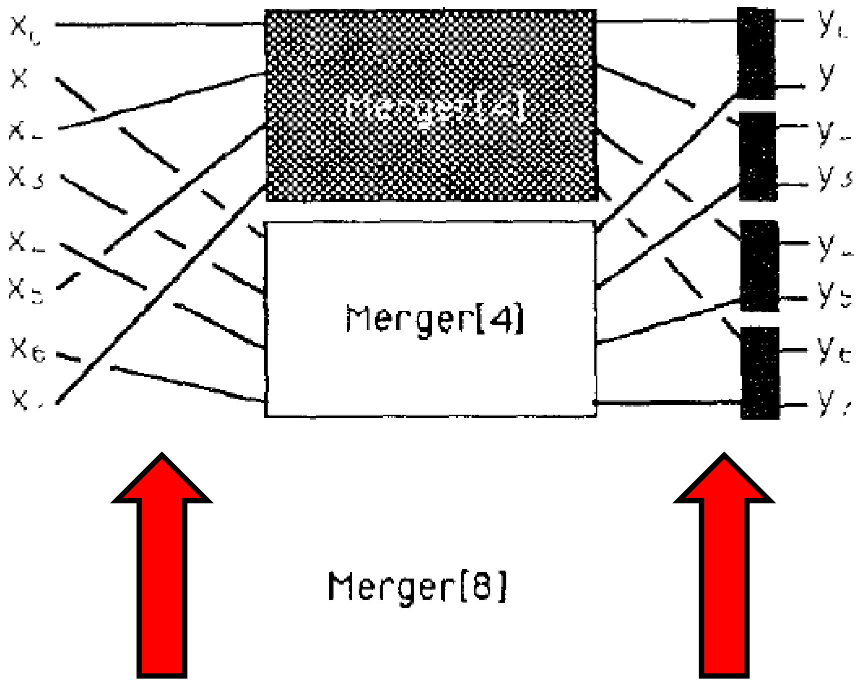
Recursive definition: MR(w) from two MR(w/2)



split odd/even **recursive definition** **pairwise balancers**

Upper merges even subsequence, lower merges odd subsequence!

Example: MR(8)

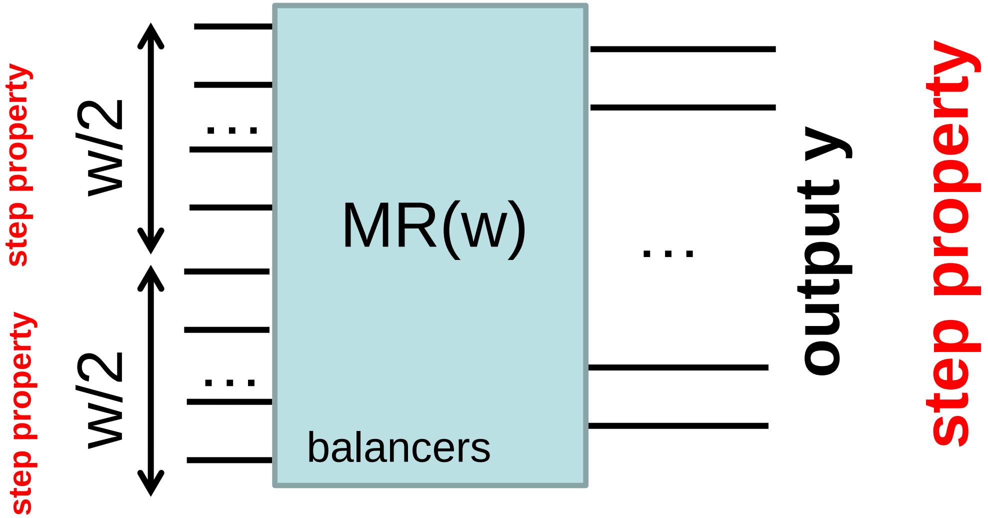


even subsequence to top
odd subsequence to bottom

top merger to upper entry
lower merger to lower entry

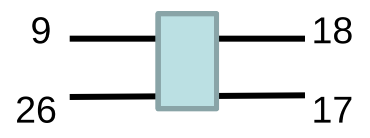
Merger MR

If MR(w) has inputs $x_0,x_{1,\dots},x_{w/2-1}$ and $x_w,x_{w+1,\dots},x_{w-1}$
With step property, then also output $y_0,y_{1,\dots},y_{w-1}$
has step property.



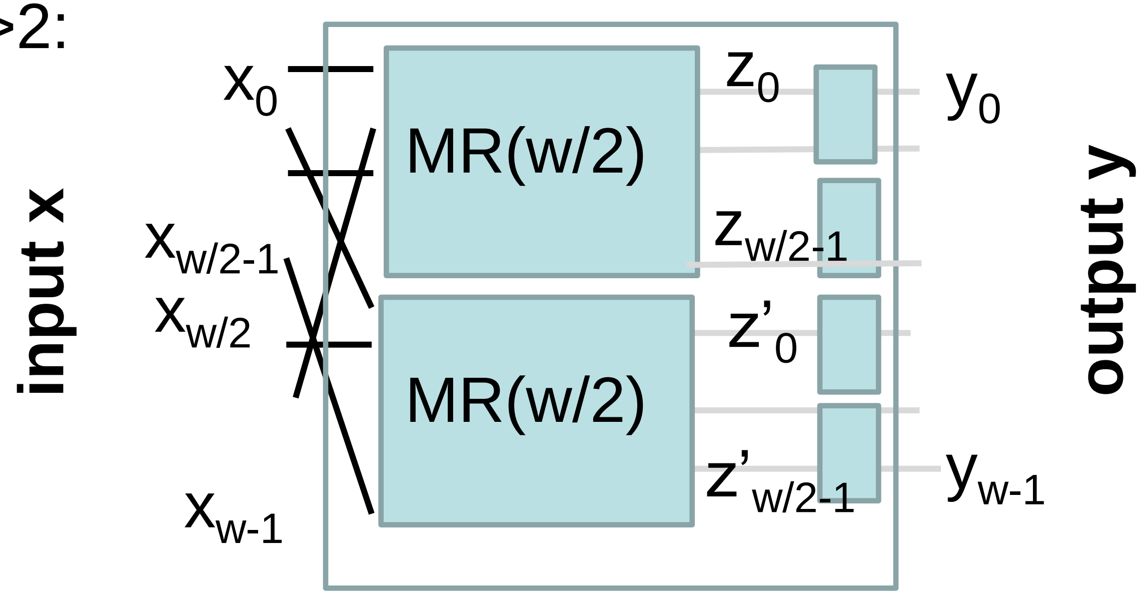
Backup: Proof by Induction

Base case w=2: Trivial as MR(2) is a balancer



Backup: Proof by Induction

Case w>2:



1. Let z_0 $z_{w/2-1}$ and z'_0 $z'_{w/2-1}$ be output of upper and lower MR(w/2)
2. Even&odd subsequences $x_0,x_1,\ldots,x_{w/2-1}$ and $x_w,x_{w+1},\ldots,x_{w-1}$ resp. must have step property too, see (B1)
3. And by Induction Hypothesis, z_0 $z_{w/2-1}$ and z'_0 $z'_{w/2-1}$ have step property
4. Let $Z = \sum z_i$ and $Z' = \sum z'_l$, so by (B2) even and odd subsequences satisfy:

$$Z = \lceil \tfrac{1}{2} \sum_{i=0}^{w/2-1} x_i \rceil + \lfloor \tfrac{1}{2} \sum_{i=w/2}^{w-1} x_i \rfloor \quad Z' = \lfloor \tfrac{1}{2} \sum_{i=0}^{w/2-1} x_i \rfloor + \lceil \tfrac{1}{2} \sum_{i=w/2}^{w-1} x_i \rceil$$

Backup: Proof by Induction

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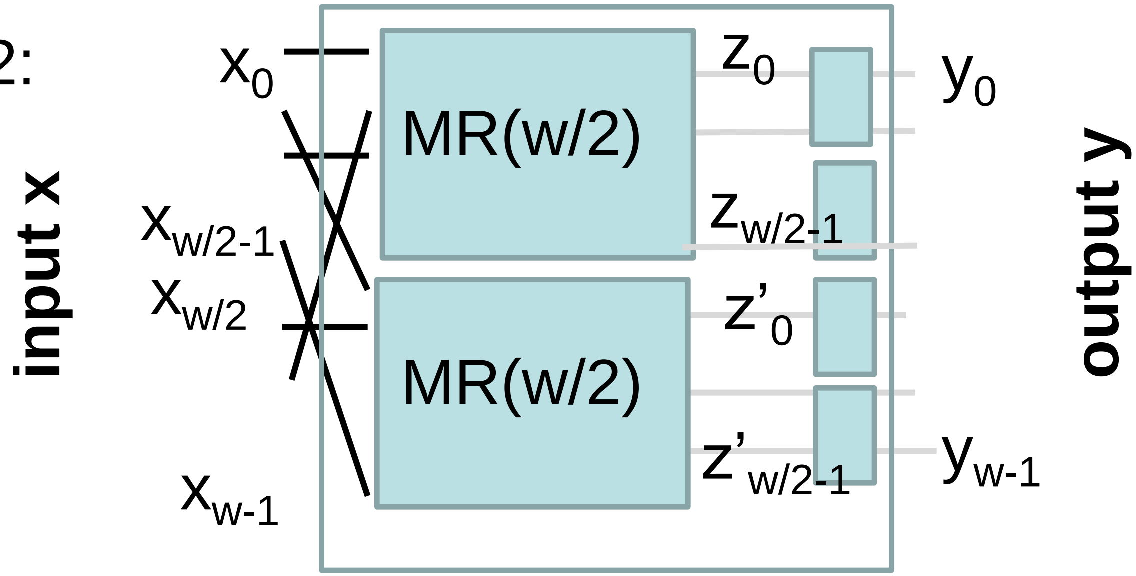

5. So Z and Z' differ by at most 1.
   - 5.a. If Z and Z' are the same, (C1) implies that $z_i = z'_i$, so output has step property too
   - 5.b. If Z and Z' differ by one, (C2) implies that there exists a unique j where z_j and z'_j differ. Let $l:=\min(z_j,z'_j)$, so output $y_i = l+1$ for $i<2j$ and $y_i = l$ for $i>2j+1$. The final balancer ensures that $y_{2j} = l+1$ and $y_{2j+1} = l$, so also step property.