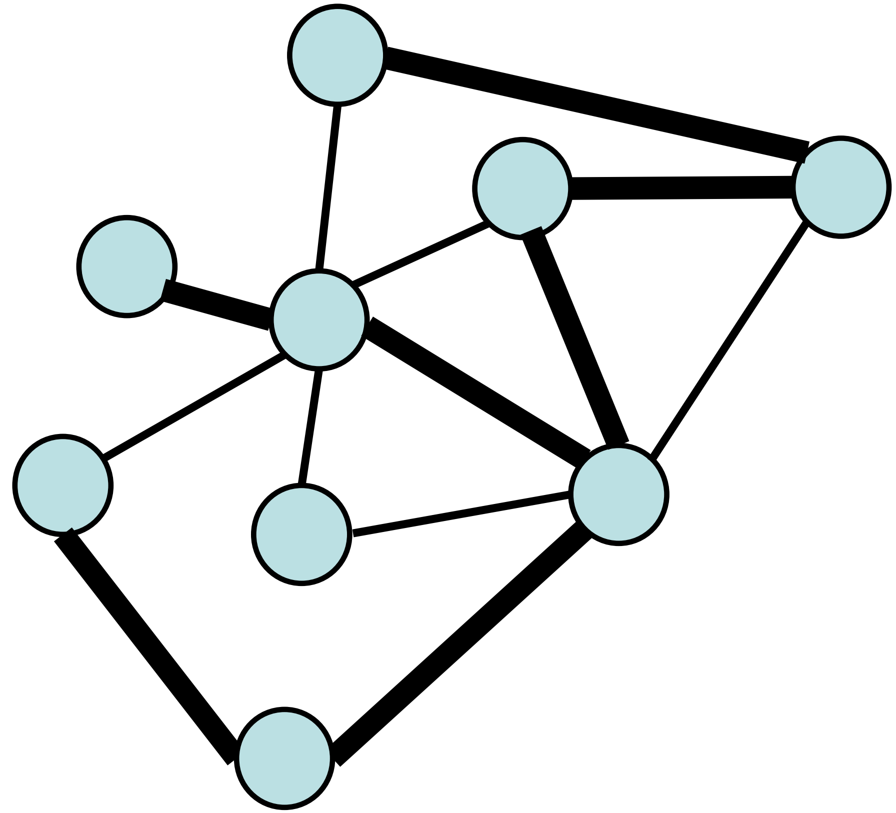
Tree Algorithms

Why (spanning) trees?

E.g., efficient broadcast, aggregation, routing, algebraic gossip...



Important trees?

E.g., breadth-first trees (BFS), minimal spanning trees (MST), ...

BFS and MST

In this lecture

BFS

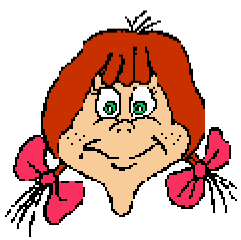
Shortest path spanning tree (unweighted) from given source.

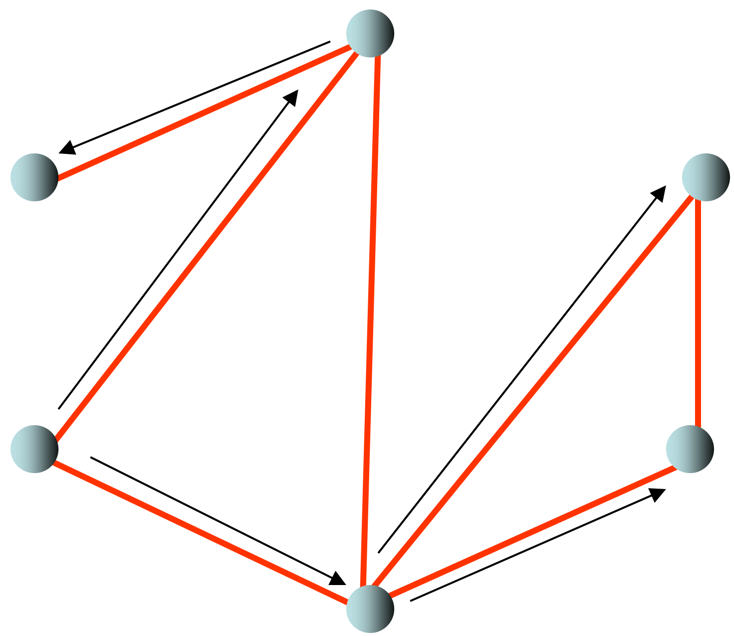
MST

Spanning tree of minimal weight.

Broadcast

Task: Send one message to all nodes.

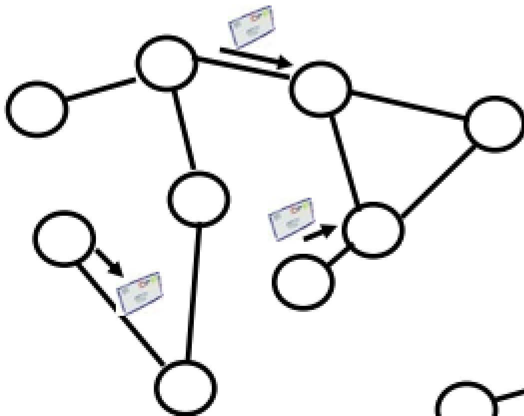




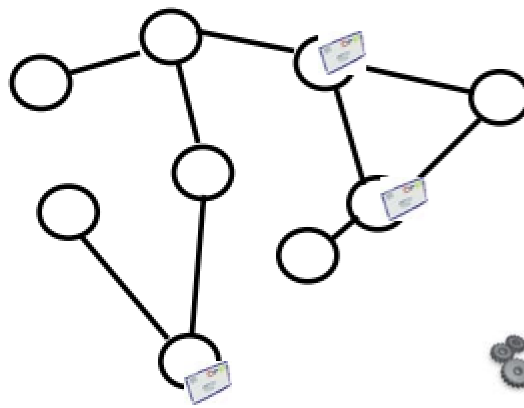
Lower bound for time and messages?

Recall: Local Algorithm

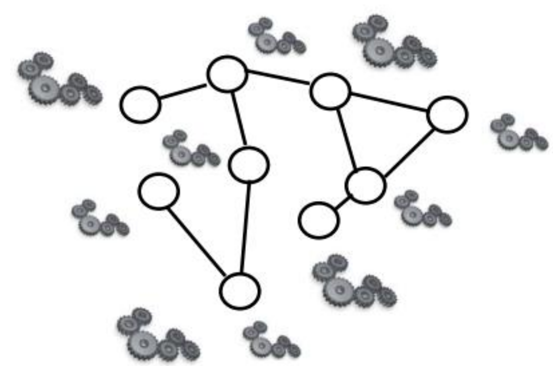
Send...



... receive...



... compute.



Broadcast

Message from one source to all other nodes.

Distance, Radius, Diameter

Distance between two nodes is # hops.
Radius of a node is max distance to any other node.
Radius of graph is minimum radius of any node.
Diameter of graph is max distance between any two nodes.

Relationship between R and D?

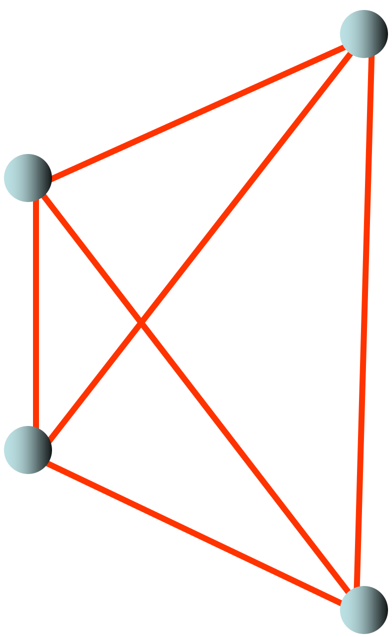
Examples....

Lemma (R, D)

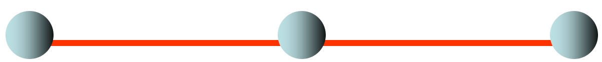
$R \leq D \leq 2R$

Where R=D?

Complete graph:

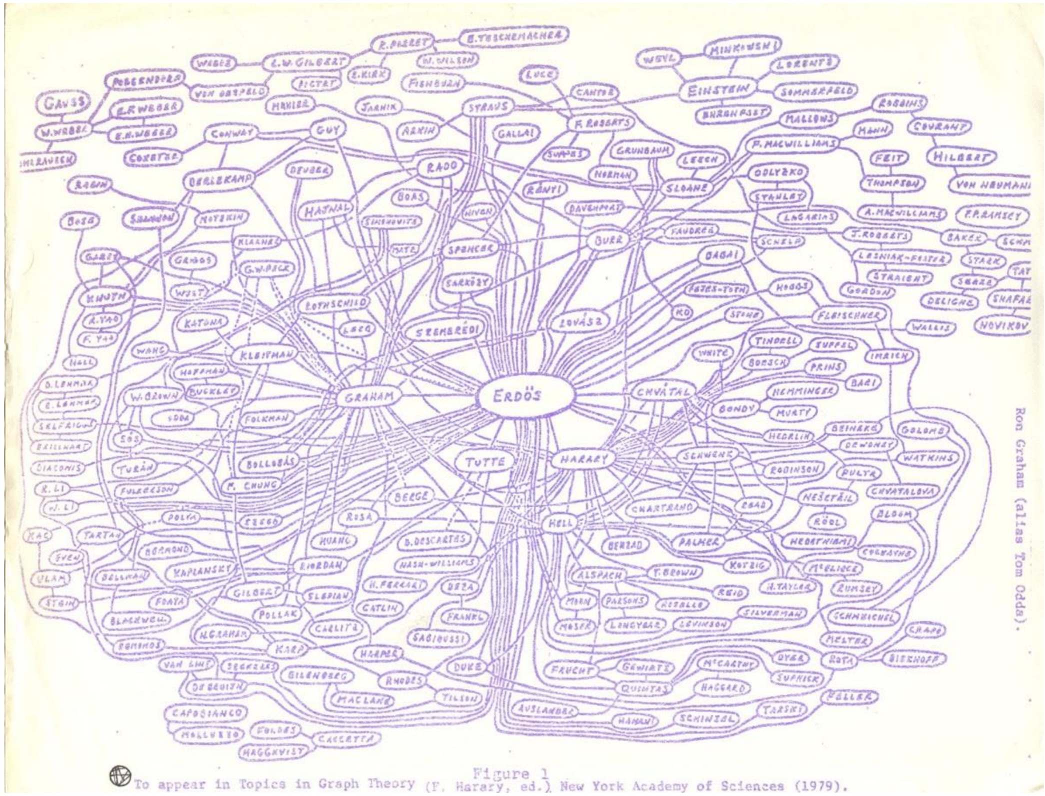


Where 2R=D?



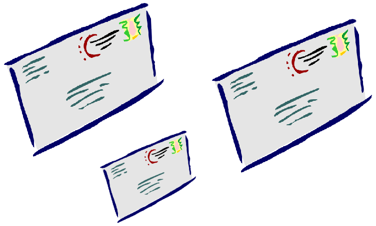
Kevin Bacon, Paul Erdös, ...

People like to find nodes of small radius in a graph! E.g., movie collaboration (link = act in same movie) or science (link = have paper together)!



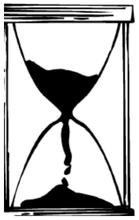
Lower Bound for Broadcast?

Message complexity?



Each node must receive message: so at least n-1.

Time complexity?



The radius of the source: each node needs to receive message.

How to achieve broadcast with n-1 messages and radius time?

Pre-computed breadth-first spanning tree...

Broadcast in Clean Networks?

Clean Graph

Nodes do not know topology.

Lower bound for clean networks?

Number of edges: if not every edge is tried, one might miss an entire subgraph!

How to do broadcast in clean network?

Flooding

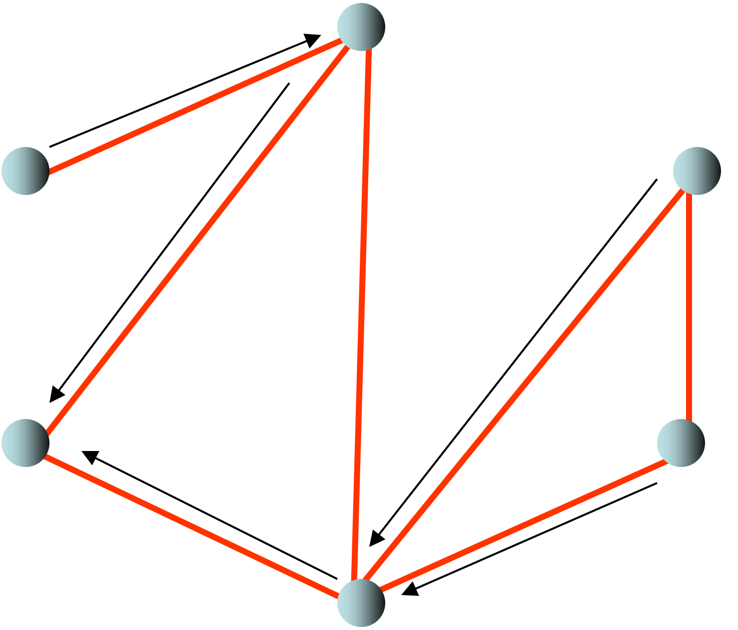
1. Source sends message to all neighbors.
2. Each other node u when receiving the message for the first time from node v (called u's parent), sends it to all (other) neighbors.
3. Later receptions are discarded.

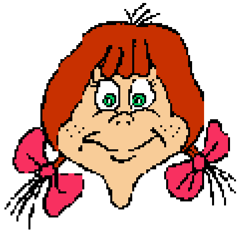
Note that parent relationship defines a tree!
In synchronous system, the tree is a breadth-first search spanning tree!

Convergecast

Convergecast

Opposite of broadcast: all nodes send message to a given node!





Purpose?

E.g., for aggregation!
E.g., find maxID!
E.g., compute average!
E.g., aggregate ACKs!

How to compute minimum efficiently?

Aggregation

Echo Algorithm

Echo Algorithm

0. Initiated by the leaves (e.g., of tree computed by flooding algo)
1. Leave sends message to its parent
2. If inner node has received a message *from each child*, it forwards message to parent

Application: convergecast to determine termination. How?

Have sub-trees completed?

Complexities?

Echo on tree, but complexity of flooding to build tree...

BFS Tree Construction

How to compute a breadth-first tree?

Flooding gives parent-relationship, but...

... breath-first only if synchronous.

How to do it in asynchronous distributed system?

Dijkstra (`link state') or Bellman-Ford (`distance vector') style

Remember the ideas?
Bellman-Ford: BGP in the Internet!

Dijkstra: grow on the „border“
Bellman-Ford: distances (distance vector)...

Asynchronous BFS Tree

Dijkstra: find next closest node („on border“) to the root

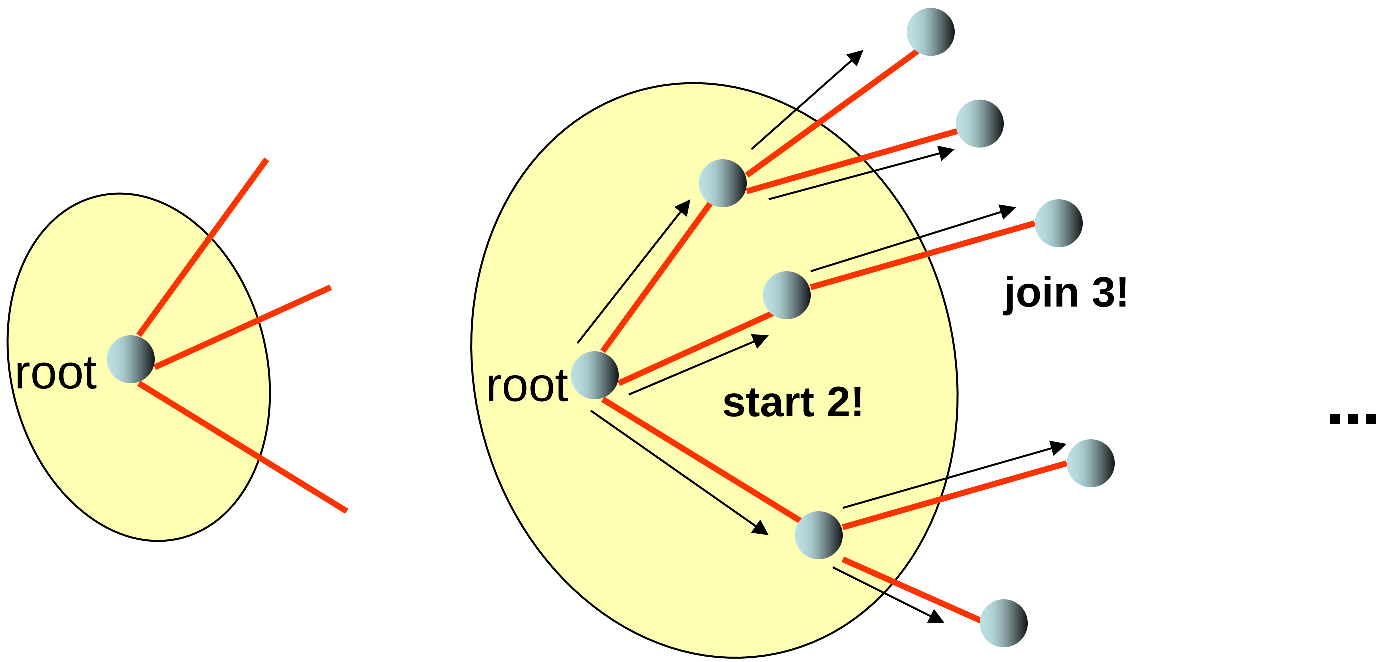
Dijkstra Style

Divide execution into *phases*. In phase p, nodes with distance p to the root are detected. Let T_p be the tree of phase p. T_1 is the root plus all direct neighbors.

Repeat (until no new nodes discovered):

1. Root starts phase p by broadcasting „**start p**“ within T_p
2. A leaf u of T_p (= node discovered only in last phase) sends „**join p+1**“ to all quiet neighbors v (u has not talked to v yet)
3. Node v hearing „join“ for first time sends back „**ACK**“: it becomes leave of tree T_{p+1}; otherwise v replied „**NACK**“ (needed since async!)
4. The leaves of T_p collect all answers and start Echo Algorithm to the root
5. Root initates next phase

Asynchronous BFS Tree: Idea



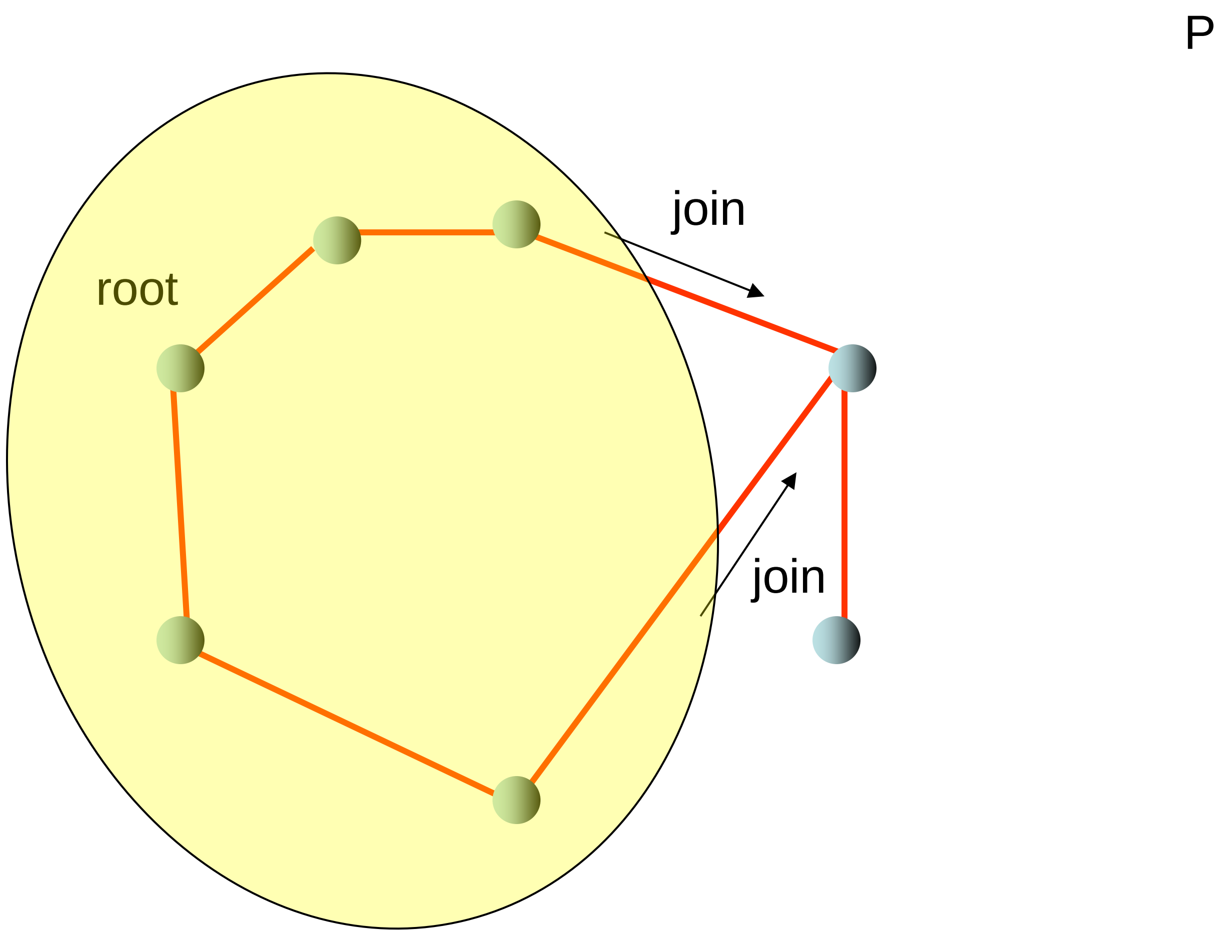
Phase 1

Phase 2

Wait until all
next hops explored...

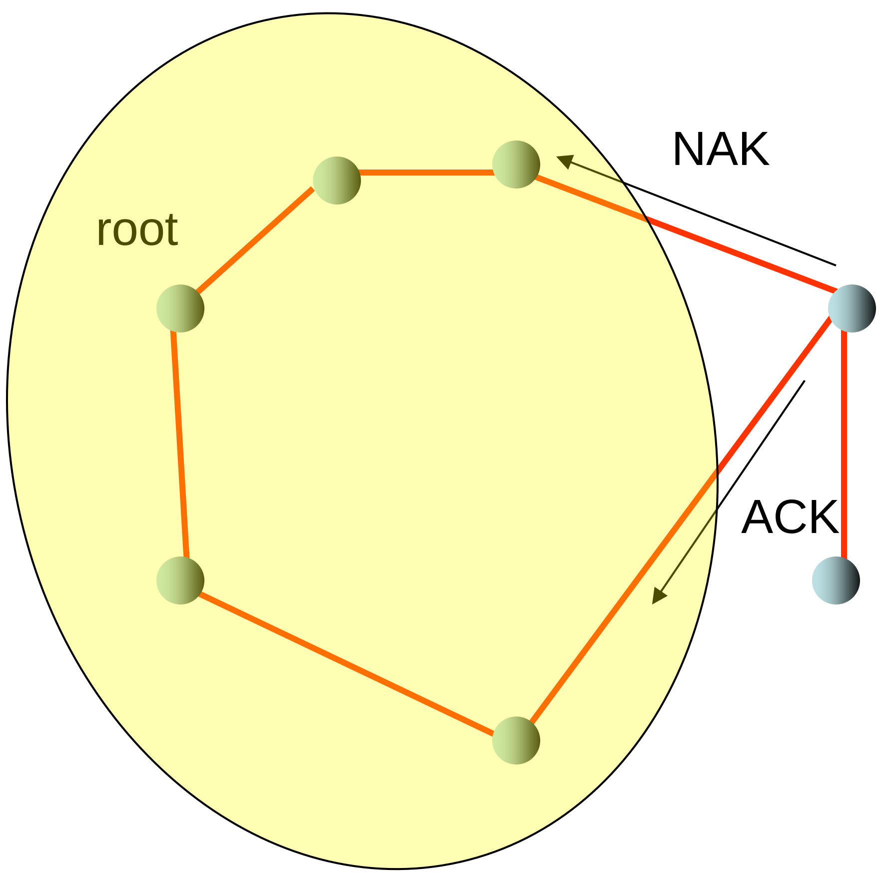
Wait until all
next hops explored...

Asynchronous BFS Tree



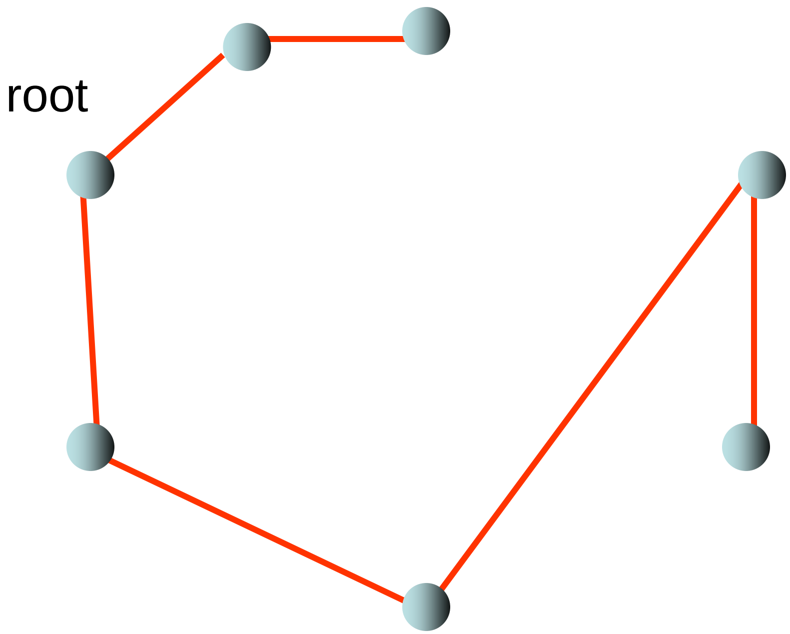
Asynchronous BFS Tree

P



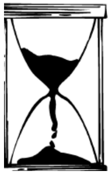
Asynchronous BFS Tree

P

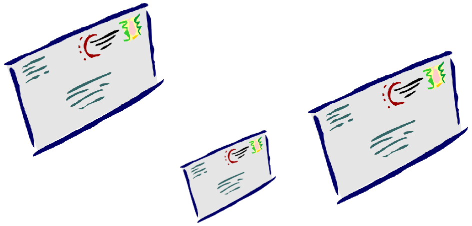


Analysis

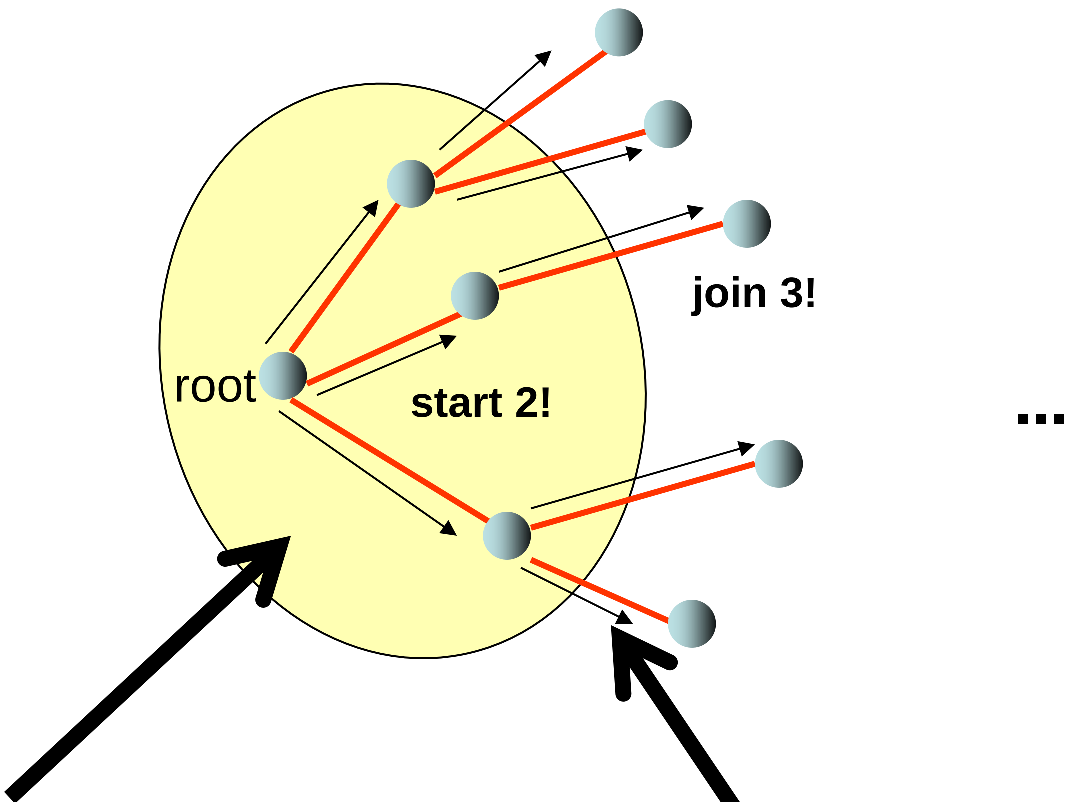
Time Complexity?



Message Complexity?



Asynchronous BFS Tree: Idea

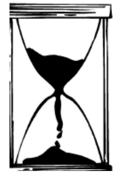


Reuse shortest path infrastructure here! Time O(D) per phase, O(D)=O(n) messages.

At most two messages per edge overall: one «join» and one «ACK/NAK»: O(m)

Analysis

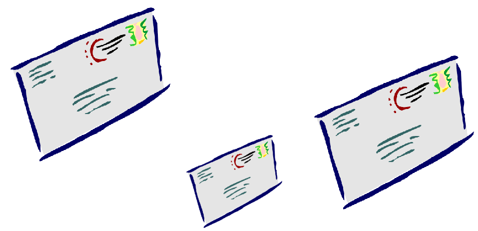
Time Complexity?



$O(D^2)$ where D is diameter of graph...

... as convergecast costs O(D), and we have D phases.

Message Complexity?



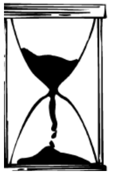
O(m+nD) where m is number of edges, n is number of nodes.

Because: Convergecast has cost O(n), one per link in tree, so over all phases O(nD). On each edge, there are at most two join messages (both directions), and there is at most an ACK/NAK answer, so +m...

Alternative algo?

Dijkstra Algorithm

Time Complexity?



$O(D^2)$ where D is diameter of graph...

... as convergecast costs O(D), and we have D phases.

Can we do it faster?! Without «back and forth»?

Asynchronous BFS Tree

Bellman-Ford: compute shortest distances by flooding an all paths; best predecessor = parent in tree

Bellman-Ford Style

Each node u stores d_u, the distance from u to the root. Initially, d_{root}=0 and all other distances are ∞. Root starts algo by sending „1“ to all neighbors.

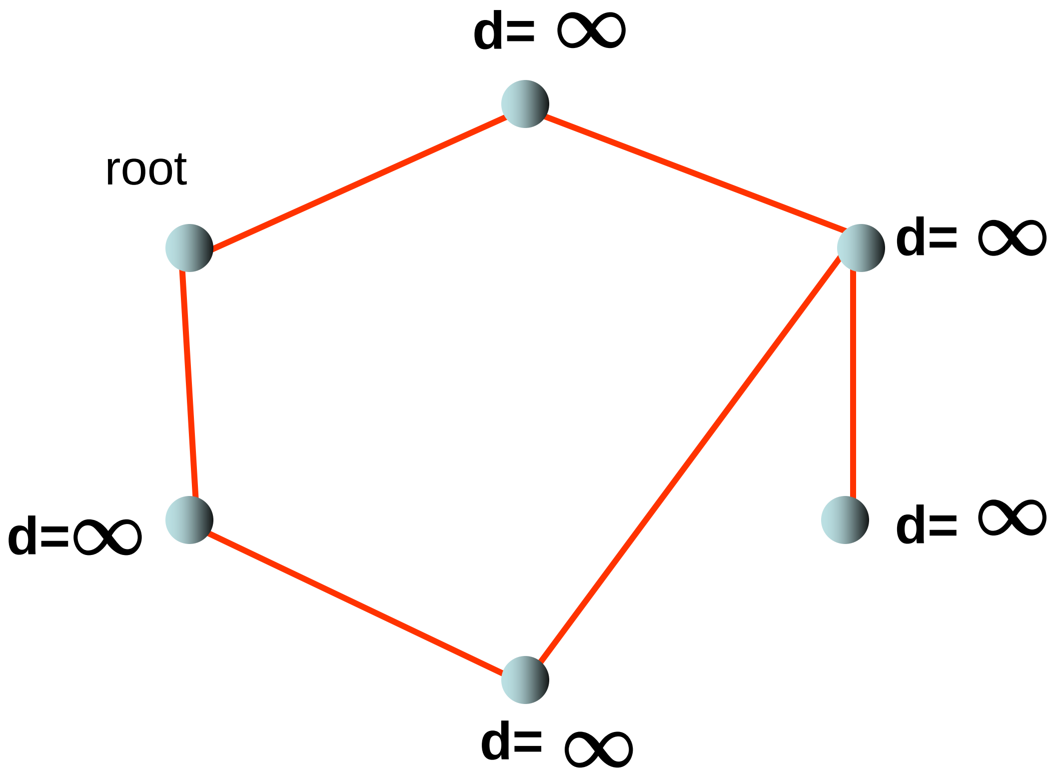
1. If a node u receives message „y“ with $y<d_u$

   $d_u := y$

   send „y+1“ to all other neighbors

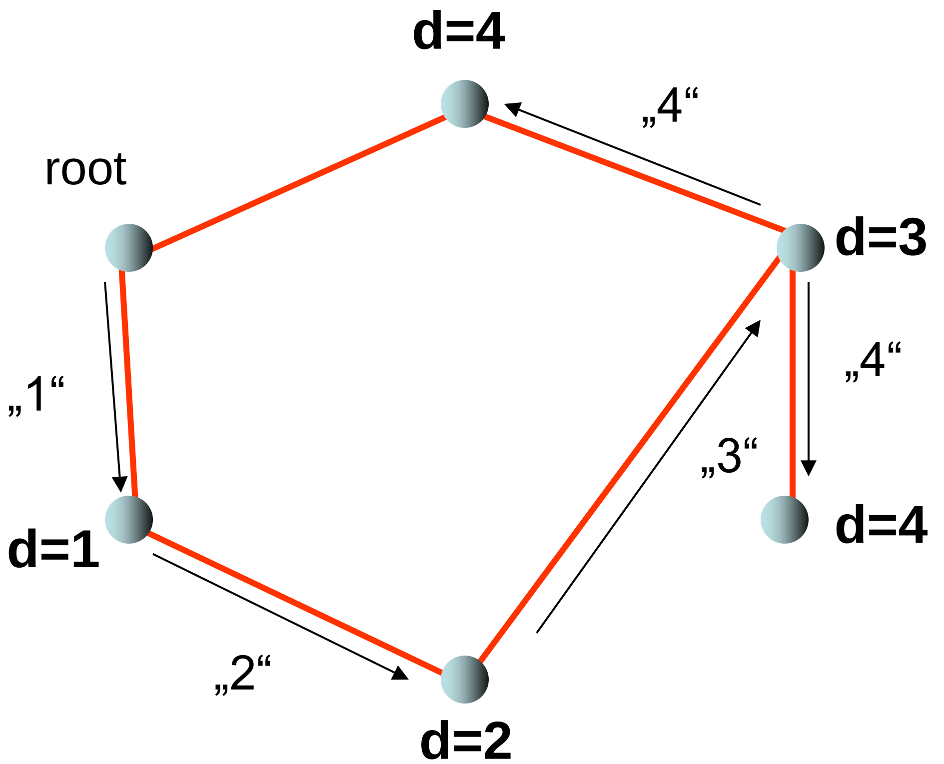
Asynchronous BFS Tree

Initially:



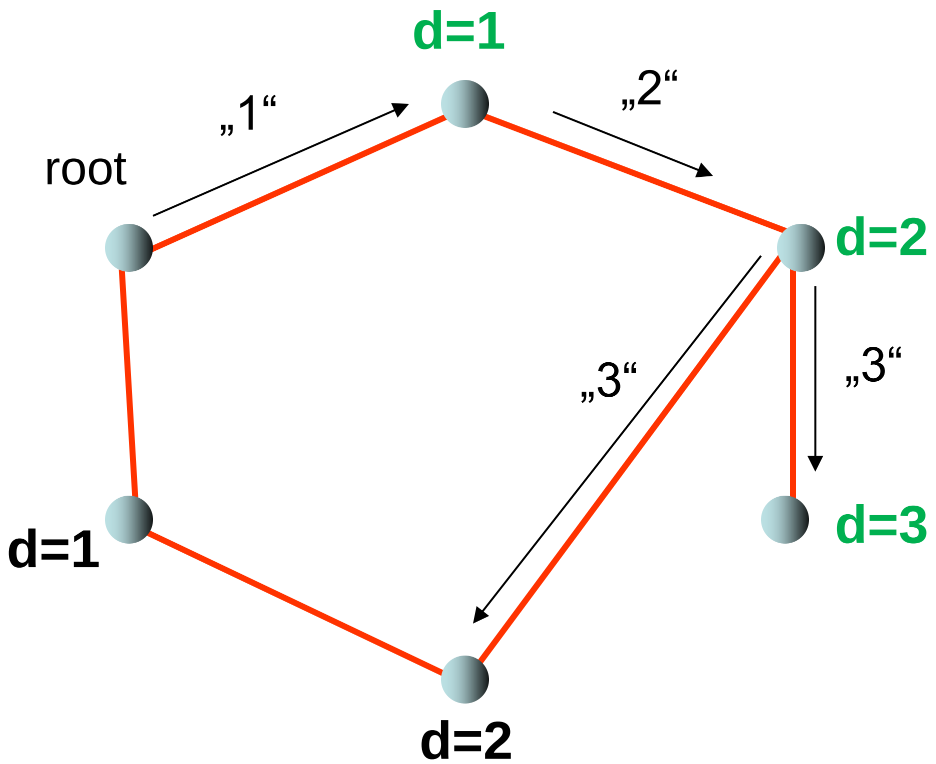
Asynchronous BFS Tree

Fast transmission:



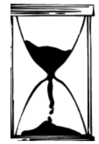
Asynchronous BFS Tree

Slow transmission:



Analysis

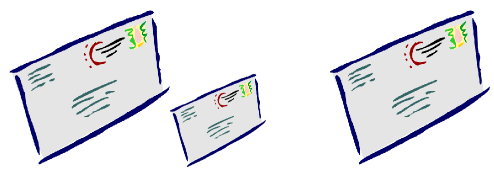
Time Complexity?



O(D) where D is diameter of graph. ☺

By induction: By time d, node at distance d got „d“.
Clearly true for d=0 and d=1.
A node at distance d has neighbor at distance d-1 that got „d-1“ on time by induction hypothesis. It will send „d“ in next time slot...

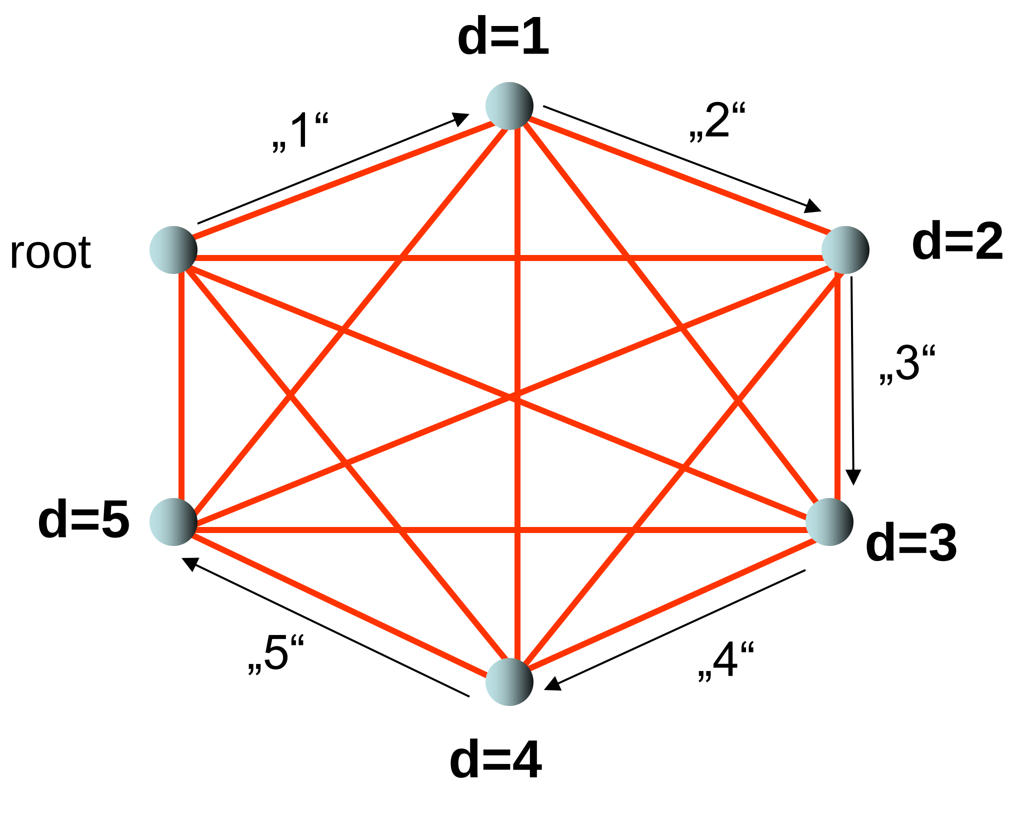
Message Complexity?



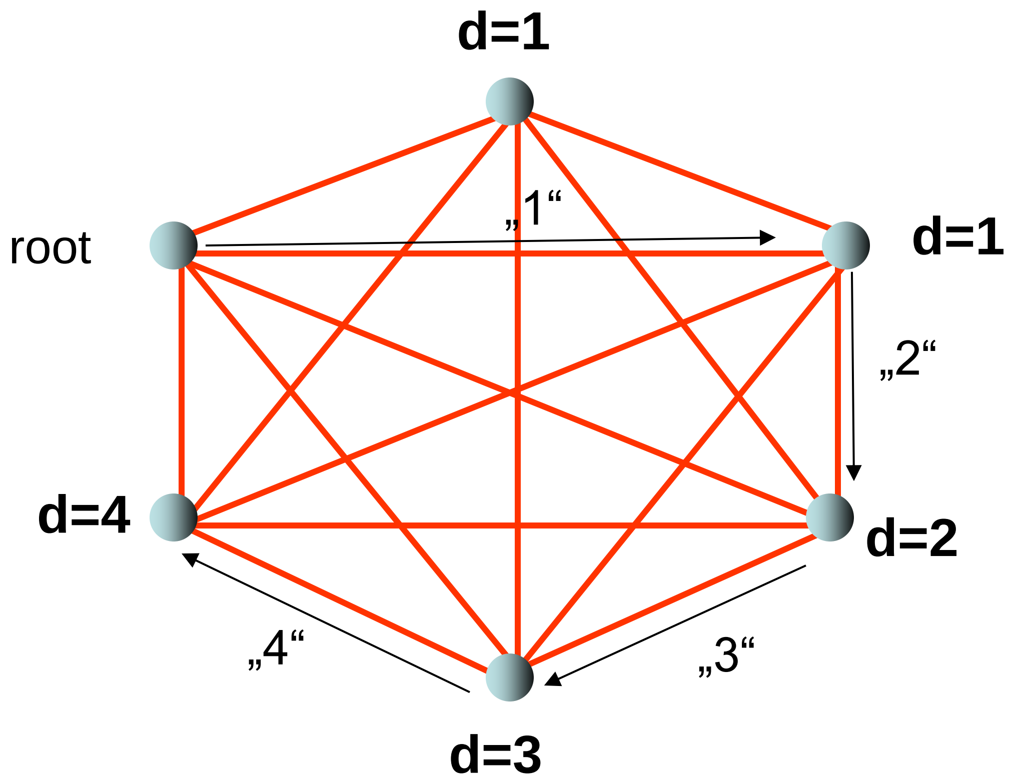
O(mn) where m is number of edges, n is number of nodes. ☹

Because: A node can reduce its distance at most n-1 times (recall: asynchronous!). Each of these times it sends a message to all its neighbors. Example?

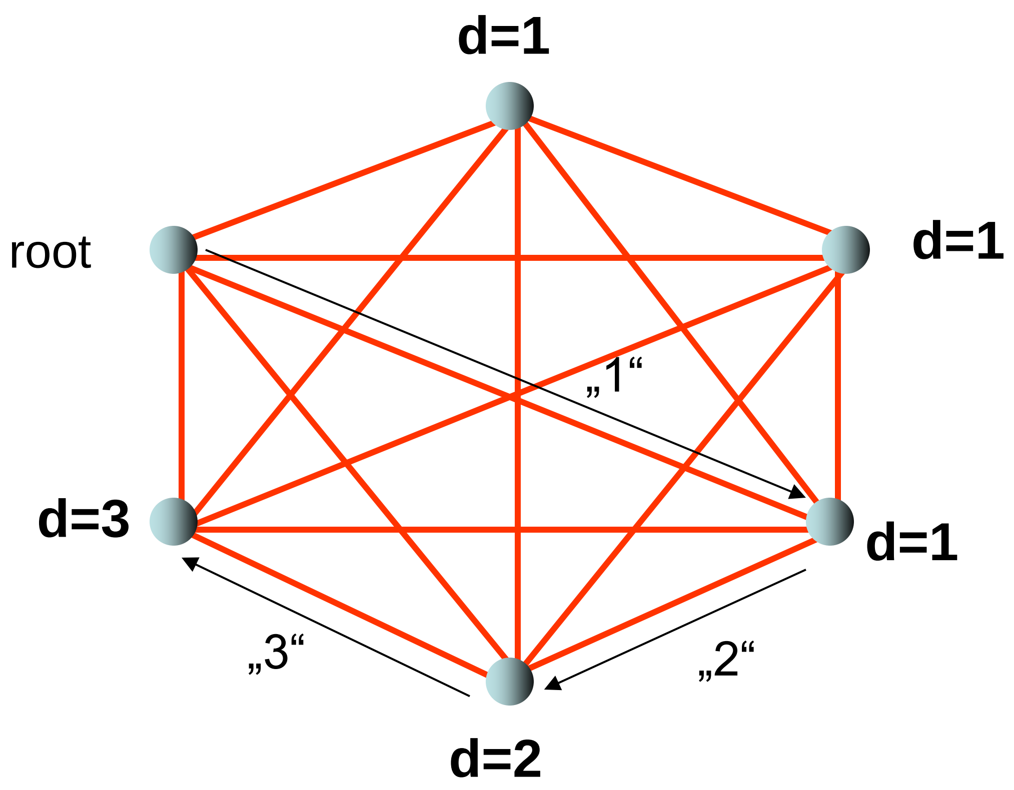
Bellman-Ford with Many Messages



Bellman-Ford with Many Messages



Bellman-Ford with Many Messages



Discussion

Which algorithm is better?

Dijkstra has better message complexity, Bellman-Ford better time complexity.

Can we do better?

Yes, but not in this course... ☺

Remark: Asynchronous algorithms can be made sychronous... (e.g., by central controller or better: local synchronizers)

MST Construction

MST

Tree with edges of minimal total weight.

Another spanning tree? Why?

For weighted graphs: tree of minimal costs...
useful building block (approximation algorithms etc.)!

Assume all links have different weights. So...
MST is unique.

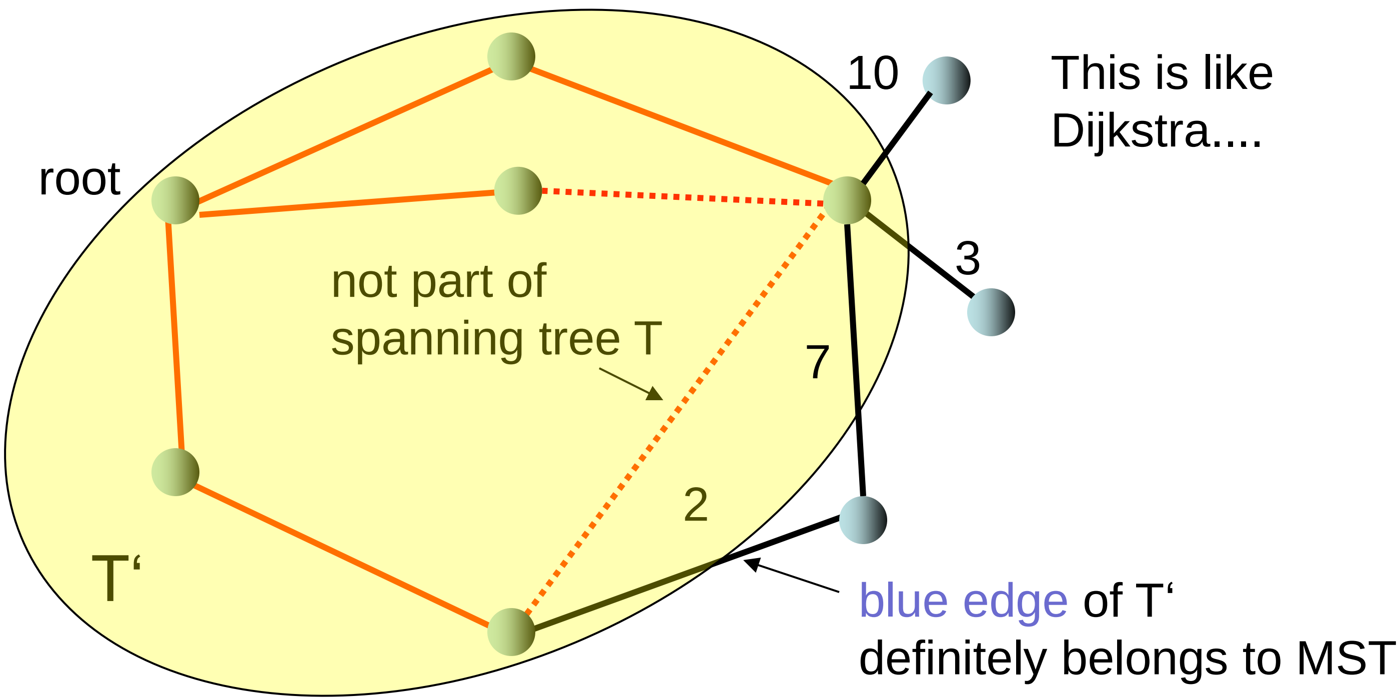
How to compute in a distributed manner (synchronously...)?! How to do it classically?

Kruskal (lightest non-cycle edge), Prim (lightest outward edge), ...

Guess: Faster or slower than BFS tree?

Blue Edge

Let T be a MST and T' a subgraph of T.
Edge e=(u,v) is outgoing edge if u $\in$ T' but v is not.
The outgoing edge of minimal weight is called *blue edge*.



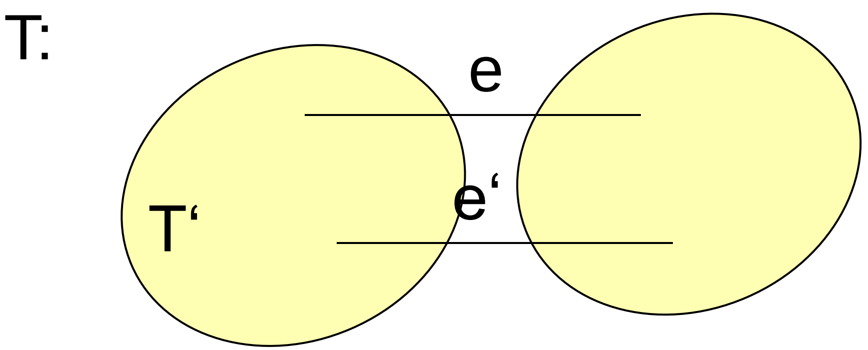
This is like Dijkstra....

Lemma

If T is the MST and T‘ a subgraph, then the blue edge of T‘ is also part of T.

Proof idea?

By contradiction! Suppose there is an other edge e‘ connecting T‘ to the rest of T. If we add the blue edge e and remove e‘ from the resulting cycle, we still have a spanning tree, but with lower cost...



So what?!

Gallager-Humblet-Spira

Ideas:

- Grow MST component by learning blue edge!
- But do many fragments in parallel!
- Each component managed by its root (the «leader»)

Distributed Kruskal

Idea: Grow components by learning blue edge!
But do many fragments in parallel!

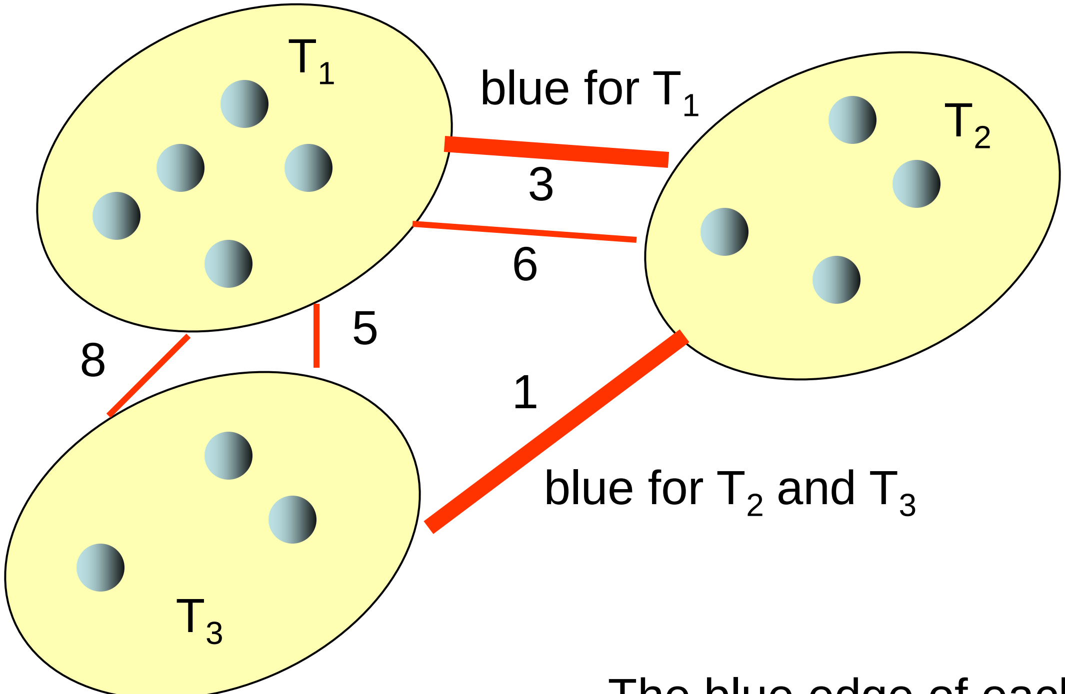
Gallager-Humblet-Spira

Initially, each node is root of ist own fragment.

Repeat (until all nodes in same fragment)

1. nodes learn fragment IDs of neighbors
2. root of fragment finds blue edge (u,v) by convergecast
3. root sends message to u (inverting parent-child)
4. if v also sent a merge request over (u,v), u or v becomes new root depending on smaller ID (make trees directed)
5. new root informs fragment about new root (convergecast on „MST“ of fragment): new fragment ID

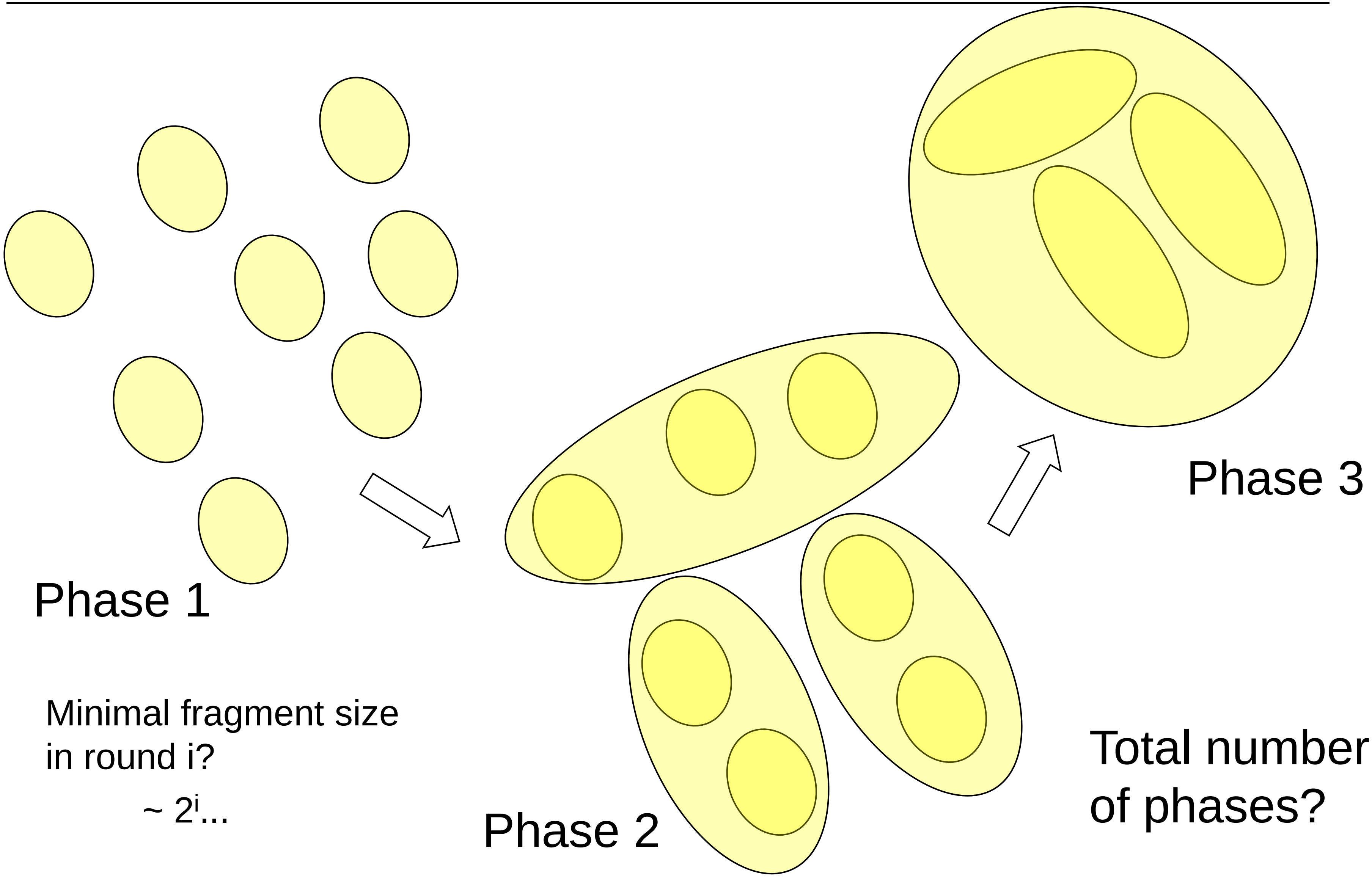
Idea: Merge Components



The blue edge of each fragment can be taken for sure: cycles not possible! (Blue edge lemma!)

So we can do it in parallel!

Idea: Components Grow Quickly

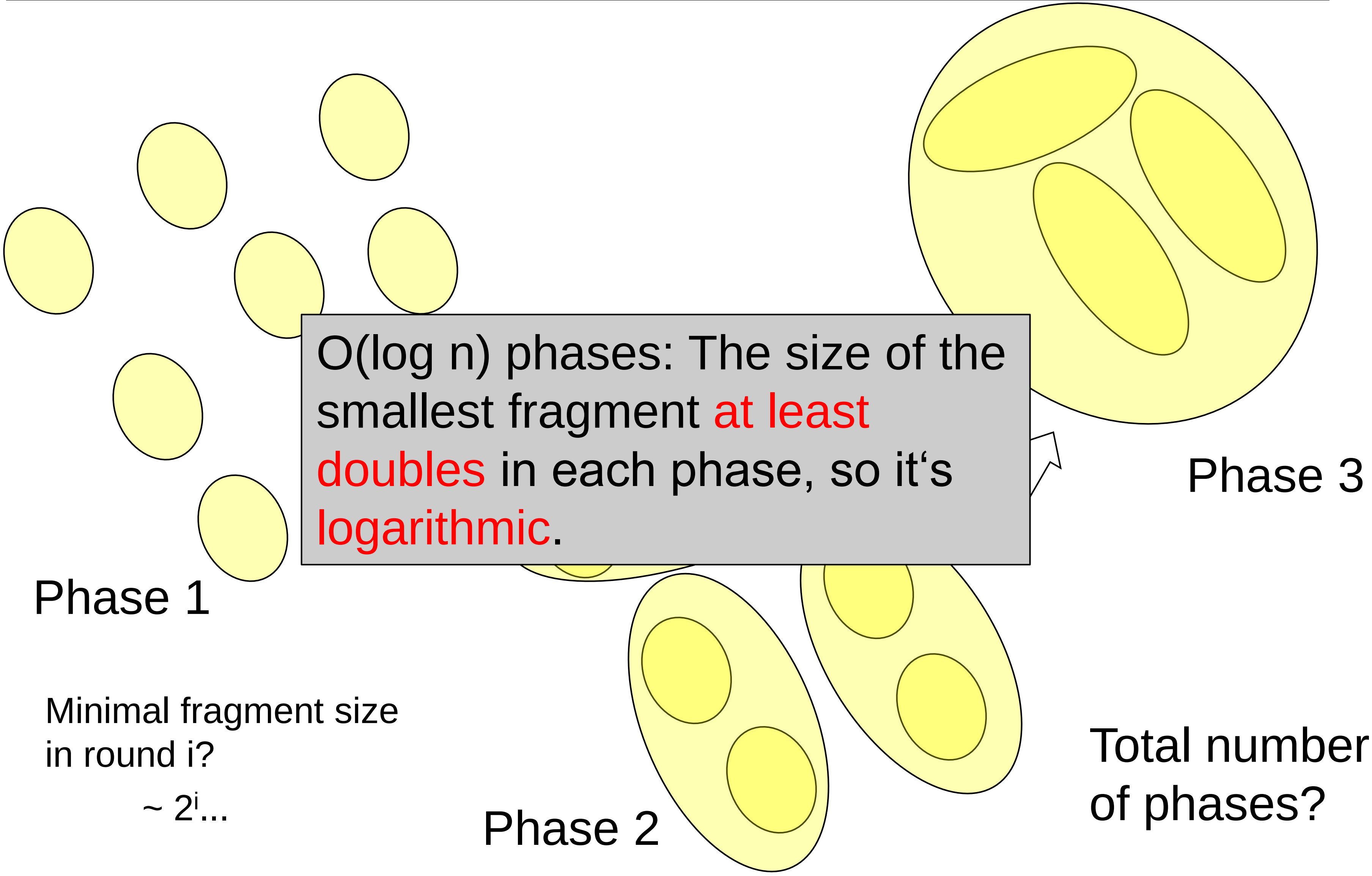


Minimal fragment size
in round i?

$\sim 2^i$...

Total number
of phases?

Idea: Components Grow Quickly



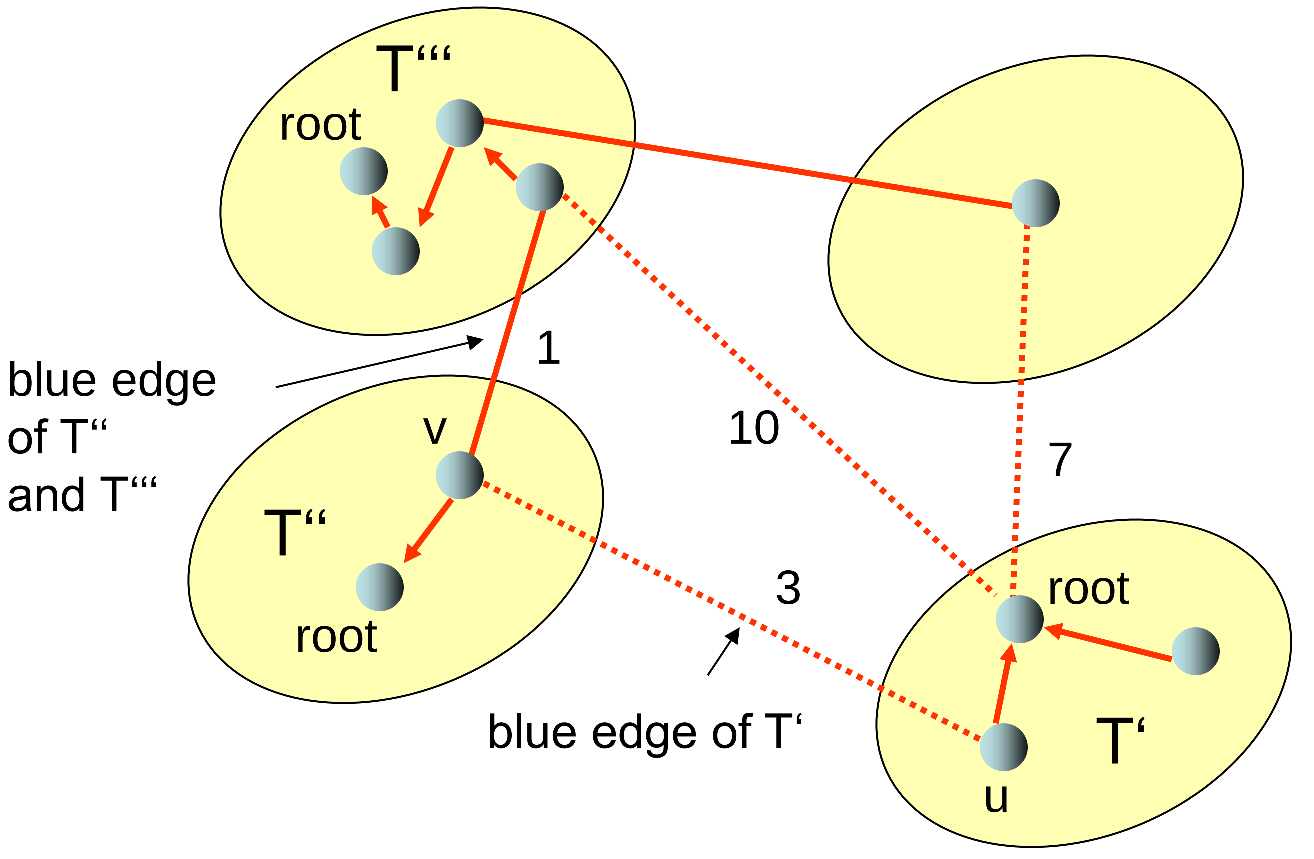
O(log n) phases: The size of the smallest fragment at least doubles in each phase, so it's logarithmic.

Minimal fragment size in round i?

~ 2^i...

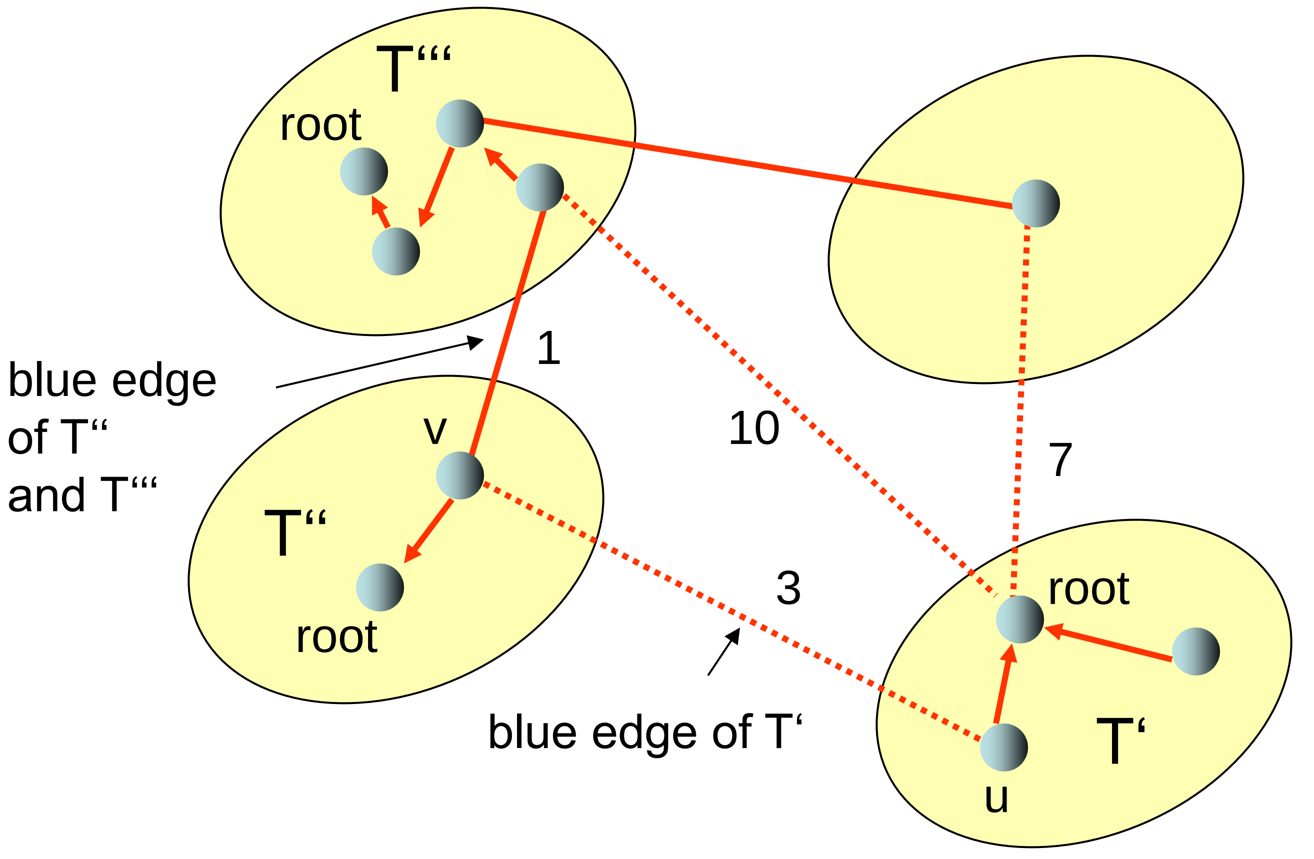
Total number of phases?

Idea: Agree on a New Root («Leader»)



Who becomes overall leader of T and T‘?
Make trees directed...

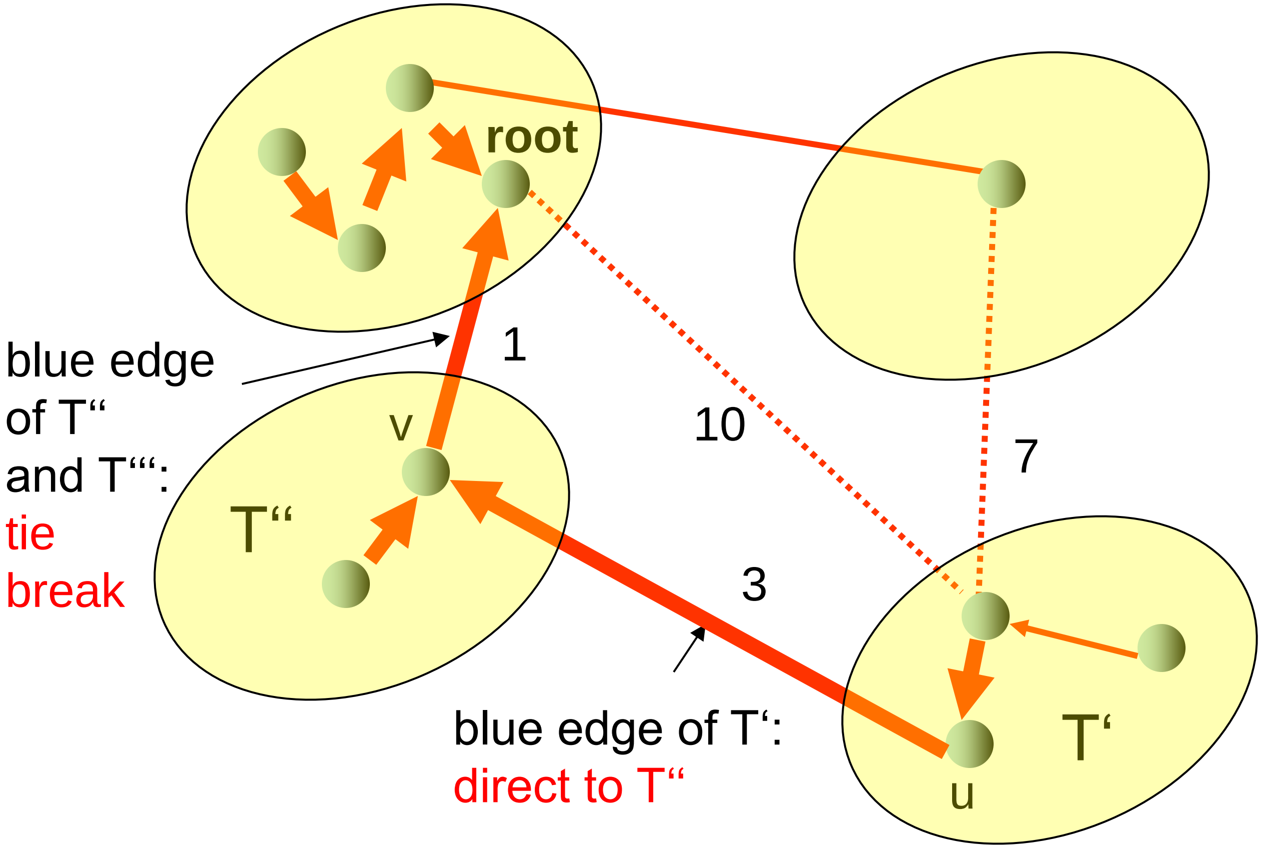
Idea: Agree on a New Root («Leader»)



All trees rooted! How to merge on blue edge (u,v)?

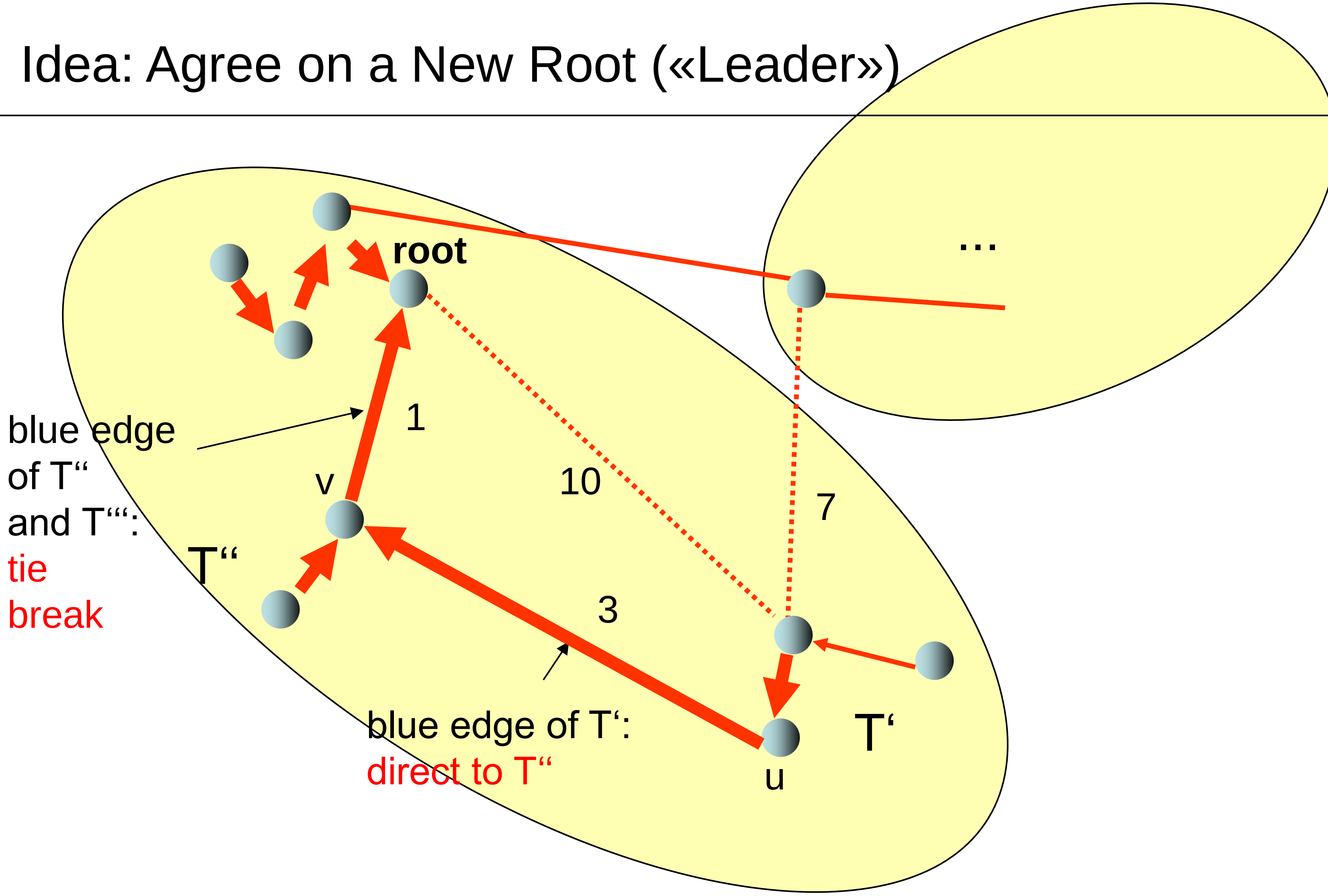
1. Invert path from root to u (u is temporary root)
2. If u sent merge request over blue edge, v becomes root; if u and v sent message over blue edge: point blue edge to smaller ID

Idea: Agree on a New Root («Leader»)



New directed tree with new root! ☺
T““ connects somewhere else...

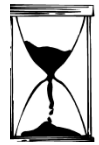
Idea: Agree on a New Root («Leader»)



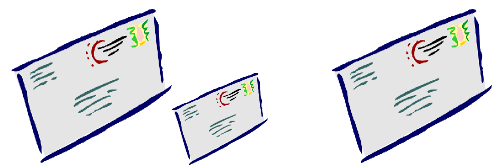
Merged fragments!

Analysis

Time Complexity?



Message Complexity?

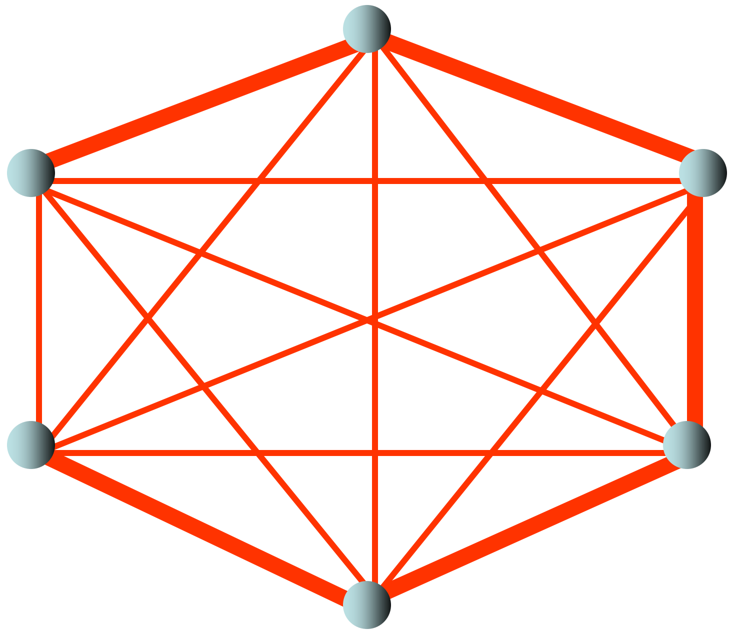


Each phase mainly consists of two convergecasts, so O(D) time and O(n) messages per phase?

Analysis

Careful:

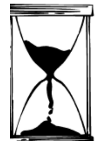
- Convergecast on MST, not on BFS tree
- MST may be larger than diameter of graph!



O(n) time for convergecast, and not O(1)...

Analysis

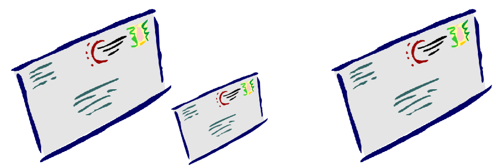
Time Complexity?



The size of the smallest fragment at least doubles in each phase, so it's logarithmic.

O(n log n) where n is graph size.

Message Complexity?



O(m log n) where m is number of edges: at most O(1) messages on each edge in a phase.

Really needed? Each phase mainly consists of two convergecasts, so O(n) time and O(n) messages. In order to learn fragment IDs of neighbors, O(m) messages are needed (in beginning of each phase: constant time).

Yes, we can do better. ☺

Discussion

- GHS solves leader election in general graphs! How?

  Last surviving root...

- Some details left out, e.g.:

  if fragment larger than other, may need to wait to find out whether neighbor also wants to merge over this edge: could do in phases (like Dijkstra BFS)

Literature for further reading:

- Peleg‘s book

End of lecture