Network Algorithms: Exercise 7
Maximal Independent Sets

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1 Deterministic Maximal Independent Sets (20 + 20 + 20 = 60 points)

In the lecture, we discussed a simple maximal independent set (MIS) algorithm in which the decisions of the nodes are based on their identifiers. The time complexity of this algorithm is $O(n)$.

We might hope that if the nodes with the largest degrees, i.e., the largest number of neighbors, decide to enter the MIS, the set of undecided nodes reduces the most. In the following algorithm we try to exploit the knowledge of the node degrees:

Assume that each node knows its degree and also the degrees of all its neighbors. If a node has a larger degree than all its undecided neighbors, it joins the MIS and informs its neighbors. Once a node v learns that (at least) one of its neighbors joined the MIS, v decides not to join the MIS.

Naturally, the algorithm does not make any progress if two or more neighboring nodes share the largest degree. As this is a difficult problem, we will assume in the following that this situation does not occur, i.e., if a node v has the largest degree, then no neighboring node has the same degree as v.[1]

a) Draw a graph that illustrates that this algorithm has a large time complexity for trees! Give a (non-trivial) lower bound on the (worst-case) time complexity for trees consisting of n nodes!

b) Construct a graph that shows that the time complexity of this algorithm is even worse for arbitrary graphs than for trees! What is the time complexity?

c) Assume that a MIS has already been constructed on a *ring*, i.e., each node knows whether it is in the independent set or not. Give an algorithm to color the ring with 3 colors in this scenario! What is the time complexity of your algorithm?

2 Inclusion-Exclusion (10 + 10 + 10 + 10 + 10 = 50 points)

In the lecture, we also discussed a faster, randomized MIS algorithm (Fast MIS 1986). In order to show the logarithmic runtime we made use of the Inclusion-Exclusion principle. The principle can be best understood from a set theory perspective (If sets describe possible events, the relationship to probability theory is immediate.). Make yourself familiar with this principle and answer the following tasks assuming three sets/probabilities A, B and C.

a) Prove the inclusion-exclusion principle graphically.

b) Show how you can use the inclusion-exclusion principle in order to give a lower bound for $|A \cup B \cup C|$.

c) Show how you can use the inclusion-exclusion principle in order to give an upper bound for $|A \cup B \cup C|$.

d) Discuss your answers on b) and c) under the assumption that the sets A, B and C are disjoint (i.e., independent).

e) Discuss briefly how your results can be extended for more than three sets.

[1] The motivation for this constraint is that if we prove that the time complexity is large even if there is no conflict in each step, then being able to break ties clearly does not help.