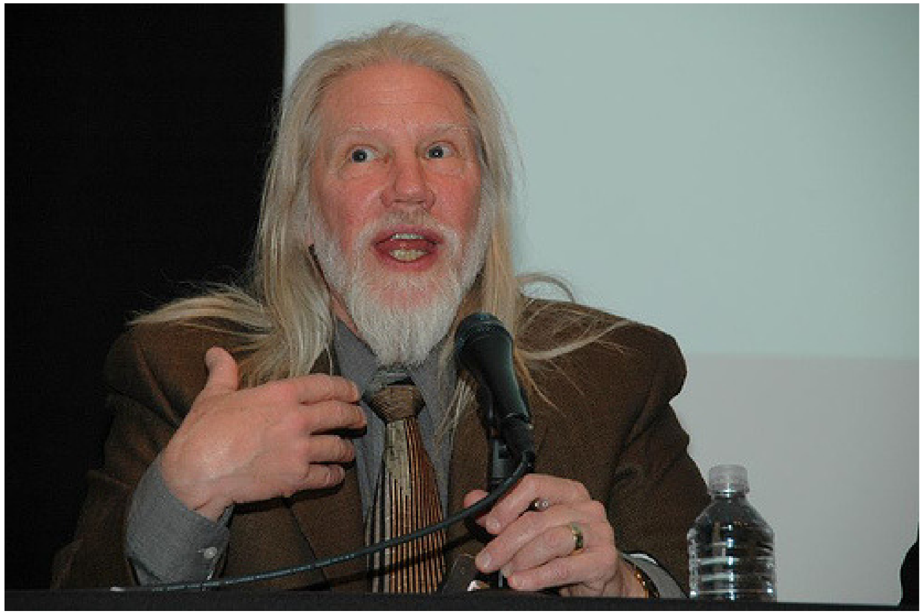


Crypto Basics 2

Public Key Cryptography

Public key exchange: Diffie-Hellmann

What is a cryptosystem?

- $K = \{0,1\}^l$
- $P = \{0,1\}^m$
- $C' = \{0,1\}^n$, $C \subseteq C'$

- $E: P \times K \rightarrow C$
- $D: C \times K \rightarrow P$

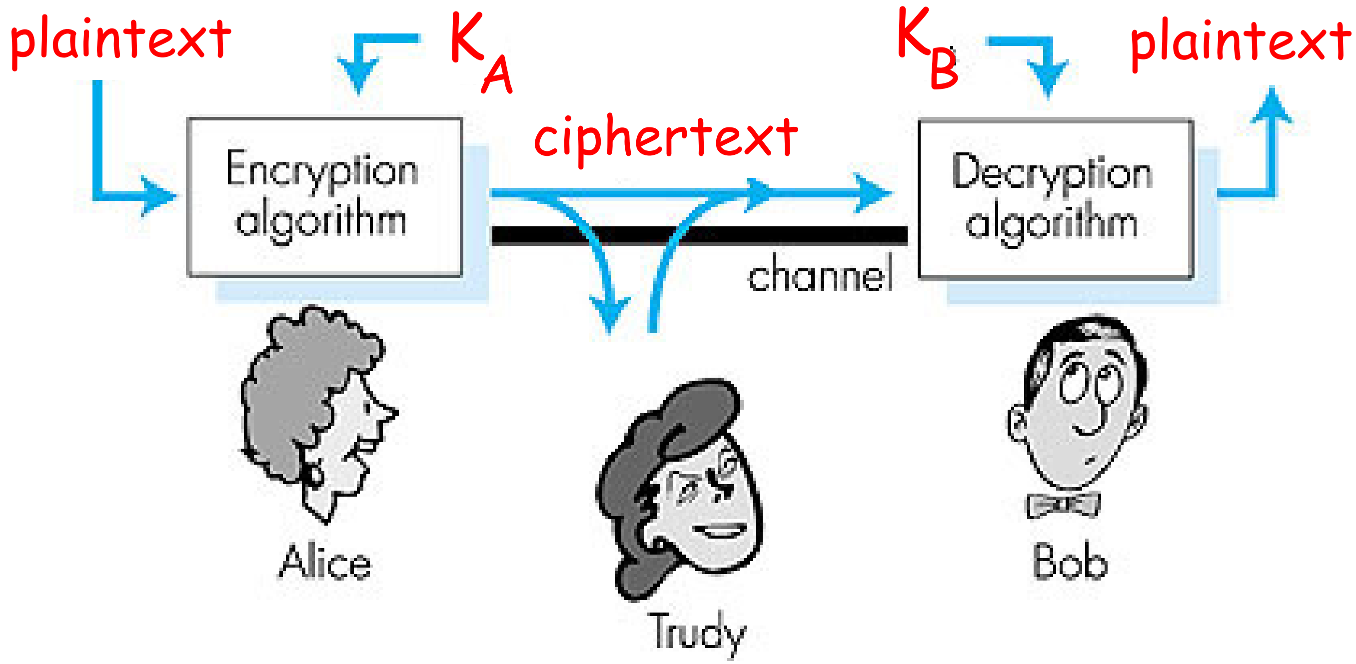
- $\forall\ p \in P,\ k \in K: D(E(p,k),k) = p$
  - It is *infeasible* to find inversion $F: P \times C \rightarrow K$

Lets start again!
This time in English … .

What is a cryptosystem?

- A pair of algorithms that take a key and convert plaintexts to ciphertexts and backwards later
  - Plaintext: text to be protected
  - Ciphertext: should appear like random
- Requires sophisticated math!
  - Do not try to design your own algorithms!

The language of cryptography



- Symmetric or secret key crypto:
  sender and receiver keys are identical and ***secret***
- Asymmetric or Public-key crypto:
  encrypt key public, decrypt key secret

Private-Key Cryptography

- traditional **private/secret/single key** cryptography uses **one** key
- shared by both sender and receiver
- if this key is disclosed communications are compromised
- also is **symmetric**, parties are equal
- hence does not protect sender from receiver forging a message & claiming is sent by sender

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses **two** keys –
  - a public & a private key
- **asymmetric** since parties are **not** equal
- uses clever application of number theoretic concepts to function
- complements **rather than** replaces private key crypto

Why Public-Key Cryptography?

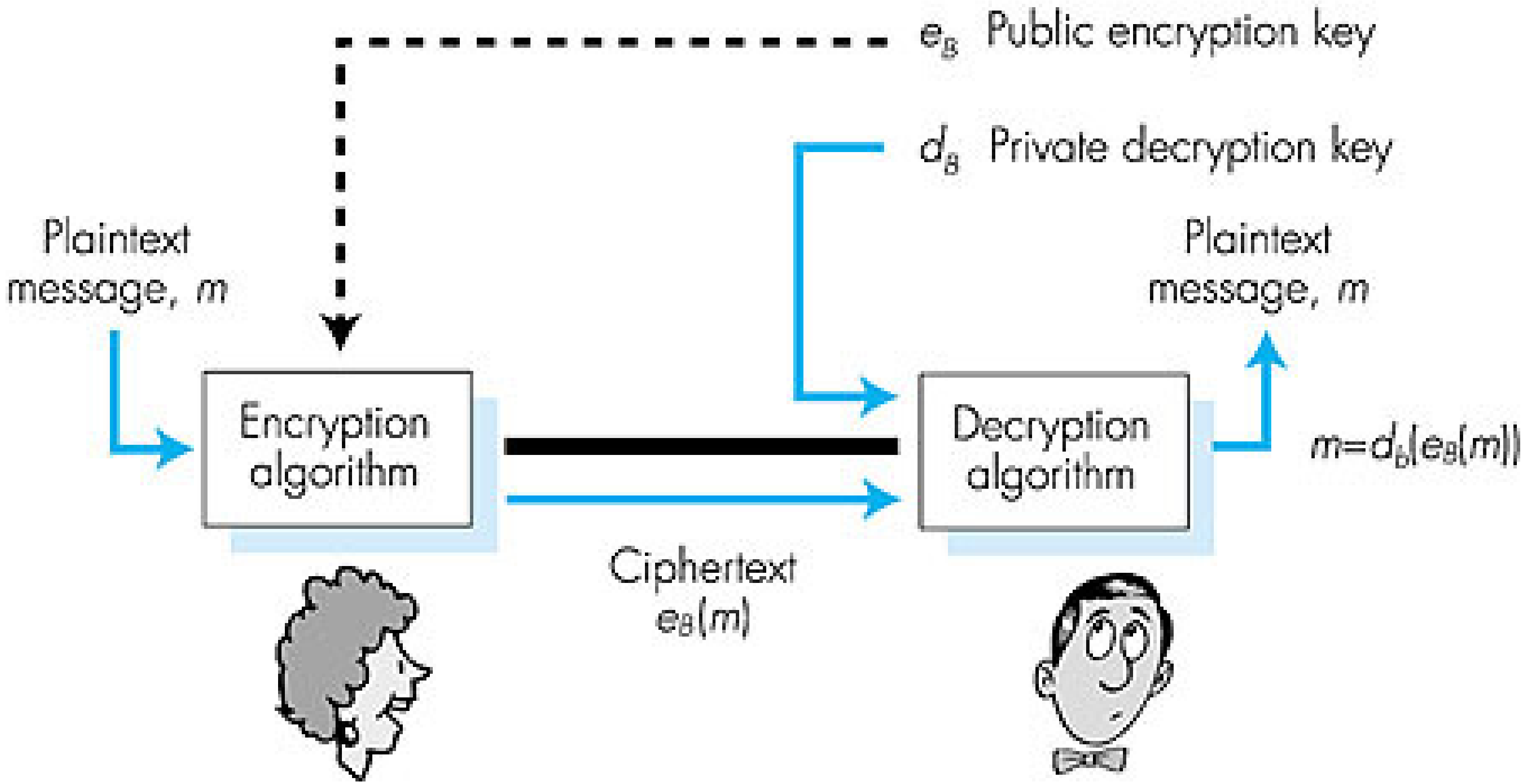
- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford Uni in 1976
  - known earlier in classified community

Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**

- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public key cryptography



Public-Key Characteristics

- Public-Key algorithms rely on two keys where:
  - it is computationally infeasible to find decryption key knowing only algorithm & encryption key
  - it is computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)

Public-Key Applications

- can classify use into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Security of Public Key Schemes

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are *too* large (>512bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (crypt-analyse) problems
- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

Public Key Cryptography

Symmetric key crypto

- Requires sender, receiver to know shared secret key
- Q: how to agree on key in first place (particularly if never "met")?
- Q: what if key is stolen?
- Q: what if you run out of keys?
- Q: what if A doesn't know she wants to talk to B?

Public key cryptography

- Radically different approach [Diffie-Hellman76, RSA78]
- Sender, receiver do *not* share secret key
- Encryption key *public* (known to *all*)
- Decryption key private (known only to receiver)
- Allows parties to communicate without prearrangement

Prime Numbers

- prime numbers only have divisors of 1 and self
  - they cannot be written as a product of other numbers
  - note: 1 is prime, but is generally not of interest
- eg. 2,3,5,7 are prime, 4,6,8,9,10 are not
- prime numbers are central to number theory
- list of prime number less than 200 is:

  2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61
  67 71 73 79 83 89 97 101 103 107 109 113 127 131
  137 139 149 151 157 163 167 173 179 181 191 193
  197 199

Relatively Prime Numbers & GCD

- two numbers `a,b` are **relatively prime** if they have **no common divisors** apart from 1
  - eg. 8 & 15 are relatively prime since factors of 8 are 1,2,4,8 and of 15 are 1,3,5,15 and 1 is the only common factor
- conversely can determine the greatest common divisor by comparing their prime factorizations and using least powers
  - eg. $300=2^1 \times 3^1 \times 5^2$ $18=2^1 \times 3^2$ hence $GCD(18,300)=2^1 \times 3^1 \times 5^0=6$

Fermat's Theorem

- $a^{p-1} = 1 \pmod{p}$
  - where p is prime and $\gcd(a,p)=1$
- also known as Fermat's Little Theorem
- also $a^p = a \pmod{p}$
- useful in public key and primality testing

Euler Totient Function ø(n)

- when doing arithmetic modulo n
- **complete set of residues** is: `0..n-1`
- **reduced set of residues** is those numbers (residues) which are relatively prime to n
  - eg for n=10,
  - complete set of residues is {0,1,2,3,4,5,6,7,8,9}
  - reduced set of residues is {1,3,7,9}
- number of elements in reduced set of residues is called the **Euler Totient Function ø(n)**

Euler Totient Function $\varnothing(n)$

- to compute ø(n) need to count number of residues to be excluded
- in general need prime factorization, but
  - for p (p prime) $\varnothing(p) = p-1$
  - for p.q (p,q prime) $\varnothing(pq) = (p-1) x (q-1)$
- eg.

  $\varnothing(37) = 36$

  $\varnothing(21) = (3-1)x(7-1) = 2x6 = 12$

Euler's Theorem

- a generalisation of Fermat's Theorem
- $a^{\varnothing(n)} = 1 \pmod{n}$
  - for any a,n where gcd(a,n)=1
- eg.

  $a=3; n=10; \varnothing(10)=4;$

  hence $3^4 = 81 = 1 \bmod 10$

  $a=2; n=11; \varnothing(11)=10;$

  hence $2^{10} = 1024 = 1 \bmod 11$

Primitive Roots

- from Euler's theorem have $a^{\varnothing(n)} \bmod n = 1$
- consider $a^m = 1 \pmod n$, $GCD(a,n)=1$
  - must exist for $m = \varnothing(n)$ but may be smaller
  - once powers reach m, cycle will repeat
- if smallest is $m = \varnothing(n)$ then a is called a **primitive root** or **generating element**
- if p is prime, then successive powers of a "generate" the group $\bmod p$
- these are useful but relatively hard to find

Discrete Logarithms

- the inverse problem to exponentiation is to find the **discrete logarithm** of a number modulo p
- that is to find `x` such that $y = g^x \pmod{p}$
- this is written as $x = \log_g y \pmod{p}$
- if g is a primitive root then it always exists, otherwise it may not, eg.
  - $x = \log_3 4 \bmod 13$ has no answer
  - $x = \log_2 3 \bmod 13 = 4$ by trying successive powers
- whilst exponentiation is relatively easy, finding discrete logarithms is generally a **hard** problem

Public-Key distribution of Secret Keys

- use previous methods to obtain public-key
- can use for secrecy or authentication
- but public-key algorithms are slow
- so usually want to use private-key encryption to protect message contents
- hence need a session key
- have several alternatives for negotiating a suitable session

Diffie-Hellman Key Exchange

- first public-key type scheme proposed
- by Diffie & Hellman in 1976 along with the exposition of public key concepts
  - note: now known that Williamson (UK CESG) secretly proposed the concept in 1970
- is a practical method for public exchange of a secret key
- used in a number of commercial products

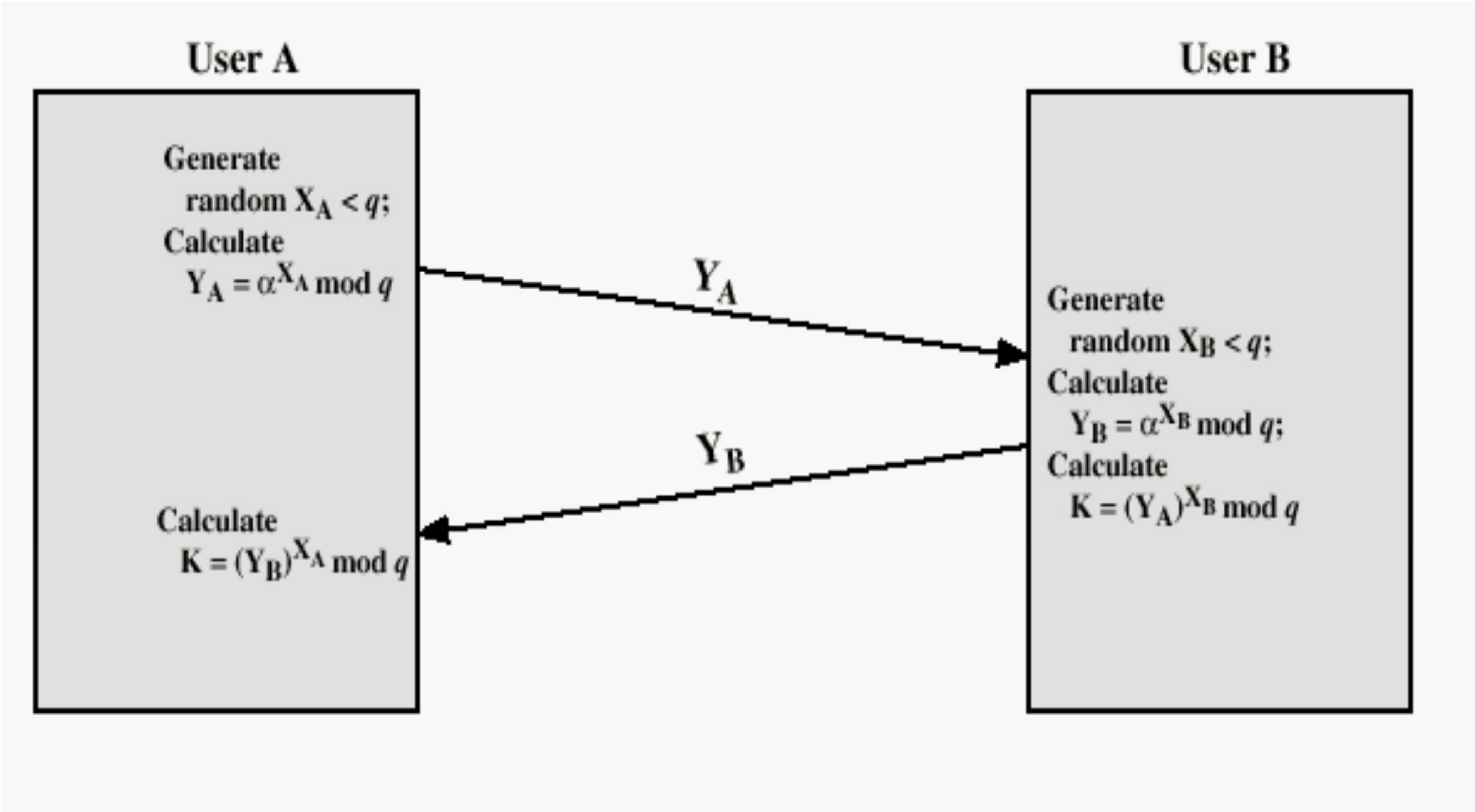
Diffie-Hellman Key Exchange

- a public-key distribution scheme
  - cannot be used to exchange an arbitrary message
  - rather it can establish a common key
  - known only to the two participants
- value of key depends on the participants (and their private and public key information)
- based on exponentiation in a finite (Galois) field (modulo a prime or a polynomial) – seems easy at first sight
- security relies on the difficulty of computing discrete logarithms (similar to factoring) – hard

Diffie-Hellman Setup

- all users agree on global parameters:
  - large prime integer q
  - a being a primitive root mod q
- each user (eg. A) generates their key
  - chooses a secret key (number): $x_A < q$
  - compute their **public key**: $y_A = a^{x_A} \bmod q$
- each user makes public that key y_A

Diffie-Hellman Key Echange



Diffie-Hellman Key Exchange

- shared session key for users A & B is K_{AB}:

  $K_{AB} = a^{x_A * x_B} \bmod q$

  $= y_A^{x_B} \bmod q$ (which **B** can compute)

  $= y_B^{x_A} \bmod q$ (which **A** can compute)

- K_{AB} is used as session key in private-key encryption scheme between Alice and Bob

- if Alice and Bob subsequently communicate, they will have the **same** key as before, unless they choose new public-keys

- attacker needs an x, thus must solve discrete log, logarithm modulo q, i.e,., compute x_A from $y_A = a^{x_A}$

Diffie-Hellman Example

- users Alice & Bob who wish to swap keys:
- agree on prime $q=353$ and $a=3$
- select random secret keys:
  - A chooses $x_A=97$, B chooses $x_B=233$
- compute respective public keys:
  - $y_A=3^{97} \bmod 353 = 40$ (Alice)
  - $y_B=3^{233} \bmod 353 = 248$ (Bob)
- compute shared session key as:
  - $K_{AB}= y_B^{x_A} \bmod 353 = 248^{97} = 160$ (Alice)
  - $K_{AB}= y_A^{x_B} \bmod 353 = 40^{233} = 160$ (Bob)

Key Exchange Protocols

- users could create random private/public D-H keys each time they communicate
- users could create a known private/public D-H key and publish in a directory, then consulted and used to securely communicate with them
- both of these are vulnerable to a meet-in-the-Middle Attack
- authentication of the keys is needed
  - Next lectures more on this!

Summary

- Have considered:
  - Principle of Public Key Cryptography
  - Number Theory basics
  - Diffie-Hellmann Key exchange