Introduction to statistics

Literature

Raj Jain: The Art of Computer Systems Performance Analysis, John Wiley

Schickinger, Steger: Diskrete Strukturen Band 2, Springer

David Lilja: Measuring Computer Performance: A Practitioner's Guide, Cambridge University Press

Goals

- Provide intuitive conceptual background for some standard statistical methods
  - Draw meaningful conclusions in presence of noisy measurements
  - Learn how to apply techniques in new situations

→ Don't simply plug and crank from a formula

- Present techniques for aggregating large quantities of data
  - Obtain a big-picture view of your results
  - Obtain new insights from complex measurement and simulation results

Statistics: Why do we need it?

1. **Aggregate data into meaningful information.**

445 446 397 226
388 3445 188 1002
47762 432 54 12
98 345 2245 8839
77492 472 565 999
1 34 882 545 4022
827 572 597 364

$$\bar{x} = \ldots$$

What is a statistic?

- "A quantity that is computed from a sample [of data]."

  Merriam-Webster

→ A single number used to summarize a larger collection of values

What are statistic*s*?

- "A branch of mathematics dealing with the collection, analysis, interpretation, and presentation of masses of numerical data."

  Merriam-Webster

→ We are most interested in analysis and interpretation here

- "Lies, damn lies, and statistics!"

The simplest statistic: a mean?

- Reduces dataset to a single number
- But what does this mean mean?
- Measures of central tendency
  - Sample mean
  - Sample median
  - Sample mode
- Other means
  - Arithmetic
  - Harmonic
  - Geometric
- Quantifying dispersion

The problem with means

- Performance is multidimensional
  - CPU or I/O time
  - Network delay
  - Interactions of various components
  - …
- Systems are often specialized
  - Performs great on application type X
  - Performs lousy on anything else
- Potentially a wide range of execution times on one system using different benchmark programs

The problem with means (2)

- Nevertheless, people still want a single number answer!
- *How to (correctly) summarize a wide range of measurements with a single value?*

Measures of central tendency

- Values that attempt to describe a set of data by identifying the "center" within that set of data
- Use this "center" to summarize overall behavior
- You will be pressured to provide "mean" value
  - Understand how to choose the best type
- Examples
  - Sample mean: "Average" value
  - Sample median: ½ of the values are above, ½ below
  - Sample mode: Most common value

Measures of central tendency (2.)

- "Sample" implies
  - Values are measured from a discrete random variable X
- Value computed is only an approximation of the true mean value of the underlying process
- True mean value cannot actually be known
  - Would require infinite number of measurements

Sample mean

- Expected value of $X = E[X]$
  - First moment of X
  - x_i = values measured ($i = \{1, \ldots, n\}$)
  - $p_i = P(X = x_i) = P(\text{we measure } x_i)$

$$E[X] = \sum_{i=1}^{n} x_i p_i$$

Sample mean (2)

- Without additional information, assume
  - p_i = constant = $1/n$ (Laplace principle)
  - n = number of measurements
- Arithmetic mean
  - Common "average"

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

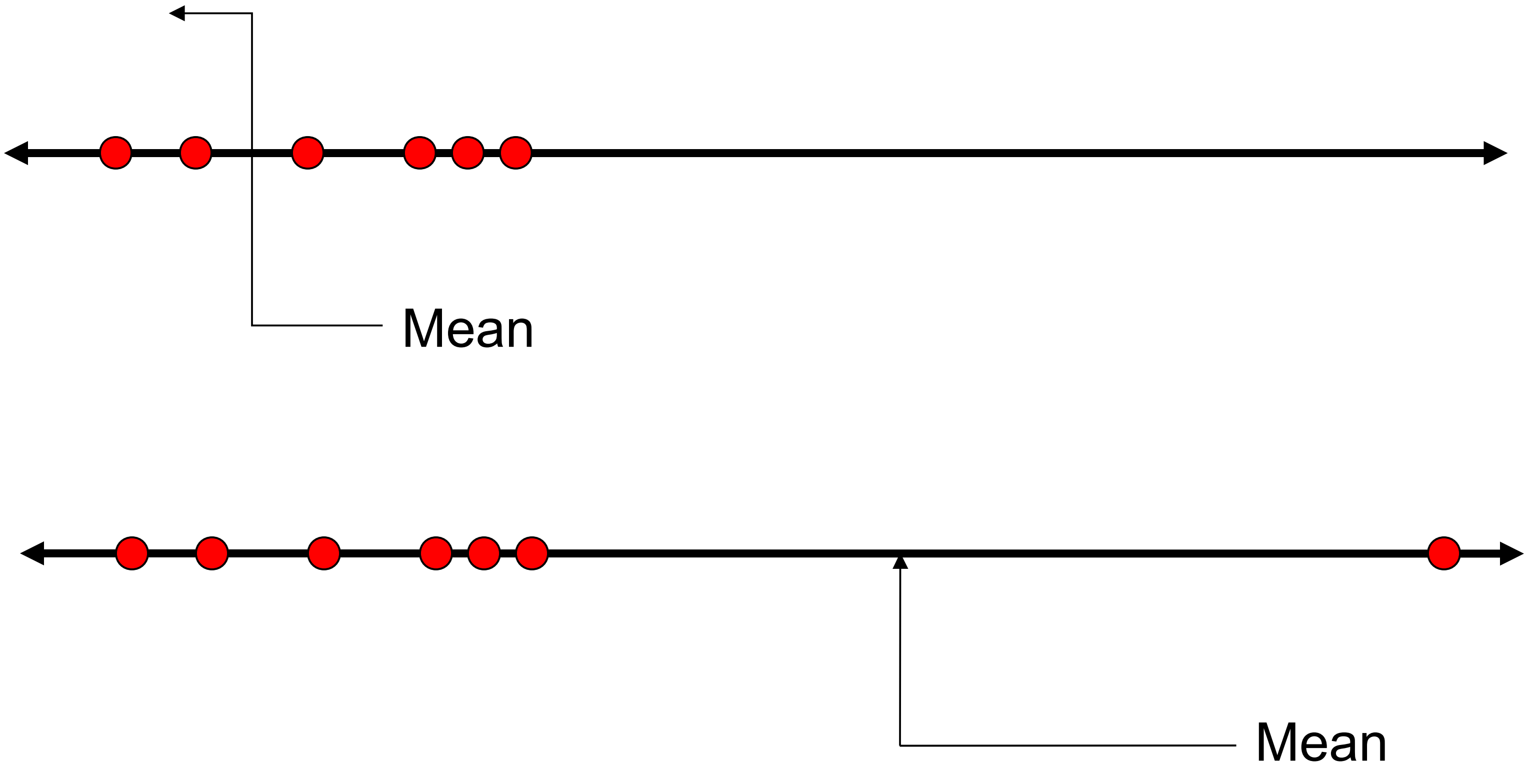
Potential problem with means

- Sample mean gives equal weight to all measured values

- Outliers can have a significant influence on the computed mean value

- May distort our intuition about the central tendency of the measured values

Potential problem with means (2.)



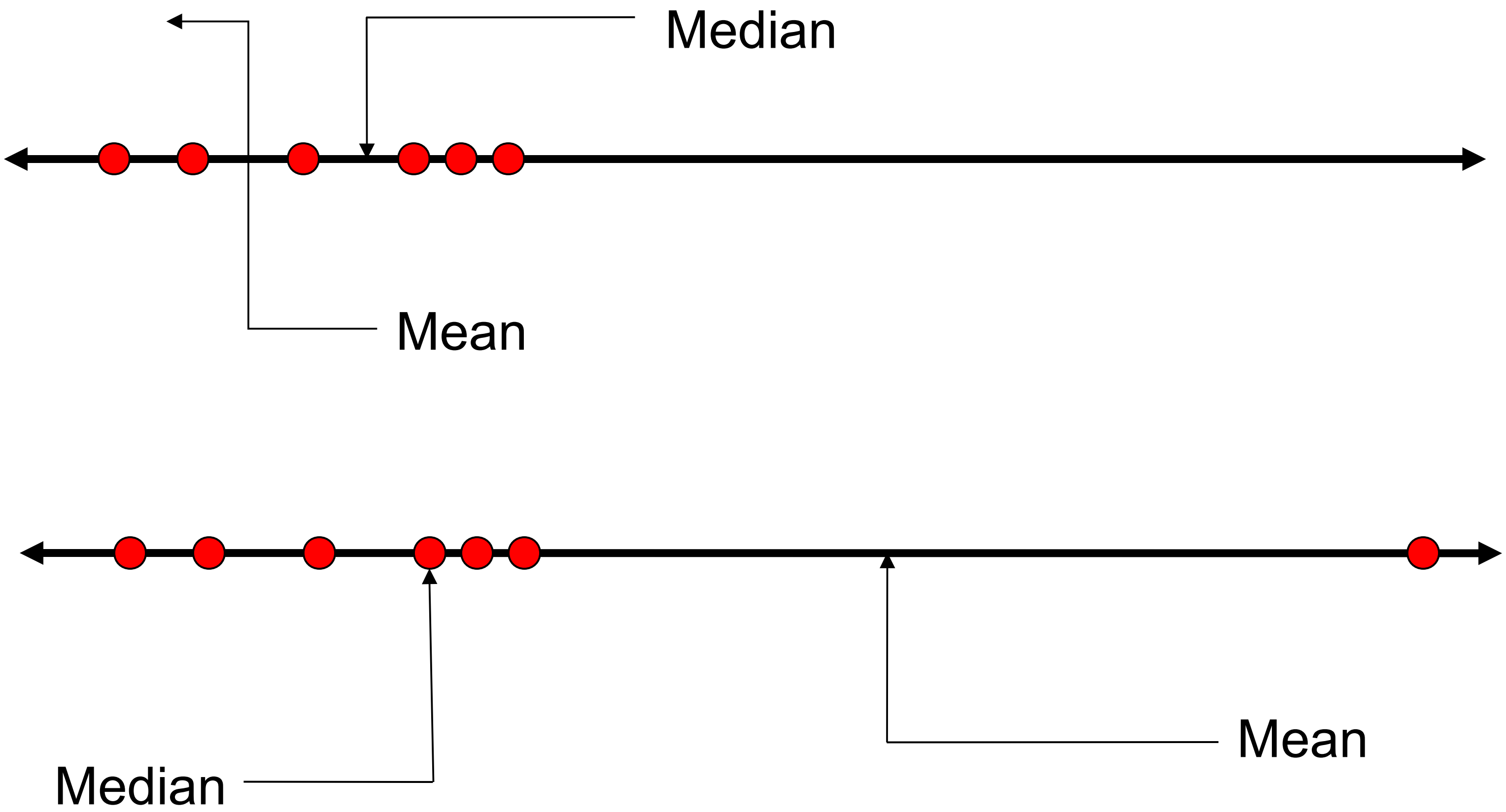
Median

- Index of central tendency with
  - ½ of the values larger, ½ smaller
  - Algorithm
    - Sort n measurements
    - If n is odd
      - Median = middle value
      - Else, median = mean of two middle values
- Reduces skewing effect of outliers

Example

- Measured values: 10, 20, 15, 18, 16
  - Mean = 15.8
  - Median = 16
- Obtain one more measurement: 200
  - Mean = 46.5
  - Median = ½ (16 + 18) = 17
- Median gives more intuitive sense of central tendency

Potential problem with means (3.)

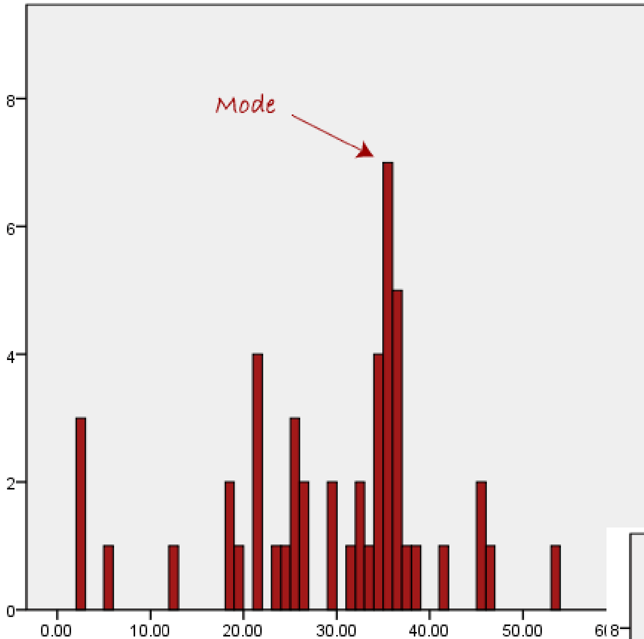


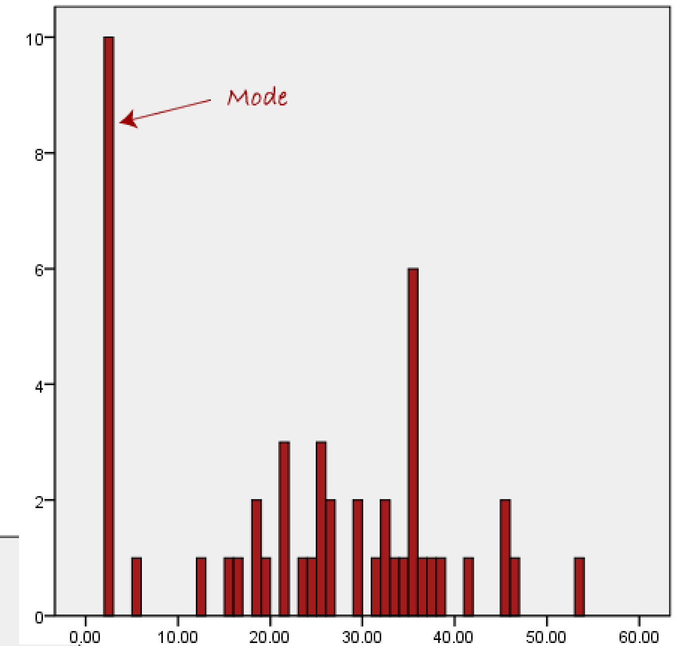
Mode

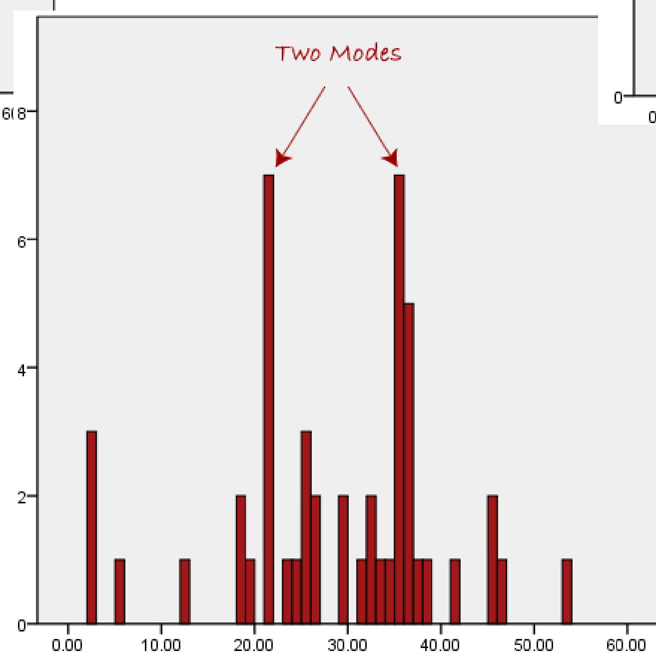
- Value that occurs most often

- May not exist
- May not be unique == multiple modes
  - E.g., “bi-modal” distribution
    - Two values occur with same frequency
- May distort our intuition about the central tendency of the measured values

Example


Mode
8
6
4
2
0
0.00
10.00
20.00
30.00
40.00
50.00


Mode
10
8
6
4
2
0
0.00
10.00
20.00
30.00
40.00
50.00
60.00


Two Modes
8
6
4
2
0
0.00
10.00
20.00
30.00
40.00
50.00
60.00

Mean, median, or mode?

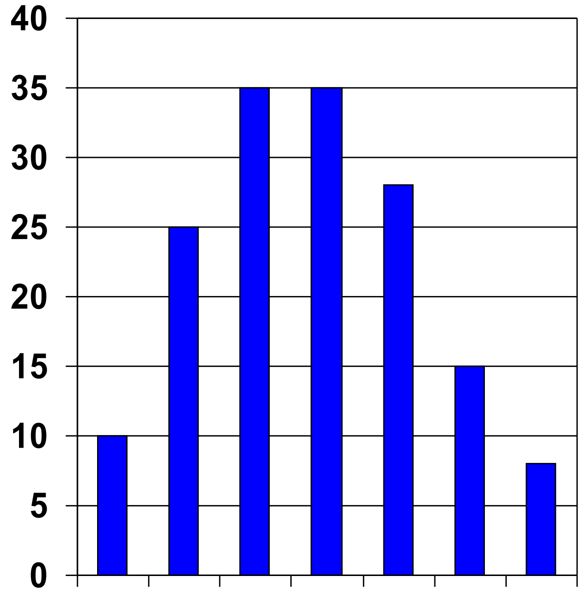
- Mean
  - If the sum of all values is meaningful
  - Incorporates all available information
- Median
  - Intuitive sense of central tendency with outliers
  - What is “typical” of a set of values?
- Mode
  - When data can be grouped into distinct types, categories (categorical data)

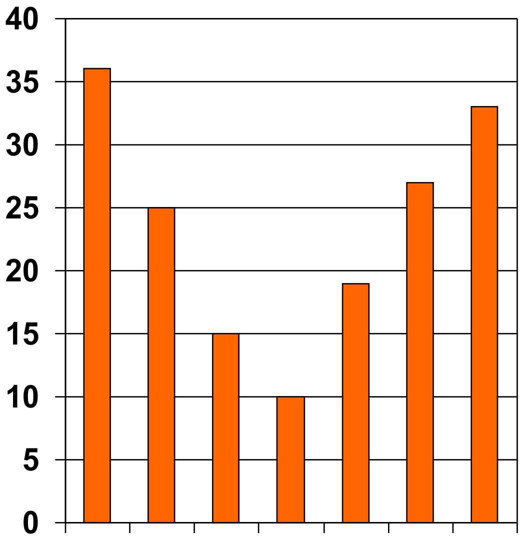
Quantifying dispersion

- How "spread out" are the values?
- How much spread relative to the mean?
- What is the shape of the distribution of values?

=> A mean hides information about variability!

Histograms





- Similar mean values
- Widely different distributions
- How to capture this variability in one number?

Measures of dispersion

Quantifies how "spread out" measurements are

- Standard deviation
- Range
  - (max value) – (min value)
- 10- and 90- percentiles
- Maximum distance from the mean
  - Max of | x_i – mean |
- Neither efficiently incorporates all available information

Determine the distribution of data?

- Plot a histogram
  - Count of observations within a cell or bucket
- Problem
  - How to determine cell size?
    - Small cells => large variations in # of obs per cell
    - Large cells => details are lost
    - Guideline: if any cell has less than five obs. increase cell size or use variable cell histogram
  - How to determine cell spacing?
    - Linear
    - Logarithmic

Determine the distribution of data(2)?

- Plot a scatter plot
  - For each value: X vs. Y
- Problem
  - Hard to visualize results in large data sets
    - Large dots => hard to distinguish points
    - Small dots => hard to see outliers

    Use two-dimensional histograms

    Use densities
  - Which scale?
    - Linear
    - Logarithmic

Determine the distribution of data(3)?

- Plot an empirical CDF
  - Concentrate $1/n$ probability at each of the n numbers in a sample
  - Describes the probability that a real-valued random variable X with a given probability distribution will be found at a value less than or equal to x

$$F_n(x) = 1/n\sum_{i=1}^{n} I(X_i \leq x)$$

- Problem
  - Tail of interest => plot CCDF

Determine the distribution of data(4)?

- Plot a density
  - Smoothed normalized counts of observations
- Problem
  - How to determine cell size?
  - How to do the smoothing
  - How to determine cell spacing?
    - Linear
    - Logarithmic

Sources of Experimental Errors

Accuracy, precision, resolution

Experimental errors

- Errors → noise in measured values
- Systematic errors
  - Result of an experimental “mistake”
  - Typically produce constant or slowly varying bias

  Controlled through skill of experimenter
  - Examples
    - Temperature change causes clock drift
    - Forget to clear cache before timing run

Experimental errors

- Random errors
  - Unpredictable, non-deterministic
  - Unbiased → equal probability of increasing or decreasing measured value
- Result of
  - Limitations of measuring tool
  - Observer reading output of tool
  - Random processes within system
- Typically cannot be controlled
  - Use statistical tools to characterize and quantify

A model of errors

- $P(X=x_i) = P(\text{to measure } x_i)$
  corresponds to the “number of possible paths”
- $P(X=x_i)$ ~ binomial distribution
- As number of error sources becomes large
  - $n \to \infty$,
  - Binomial → Gaussian (Normal)
- Thus, the bell curve

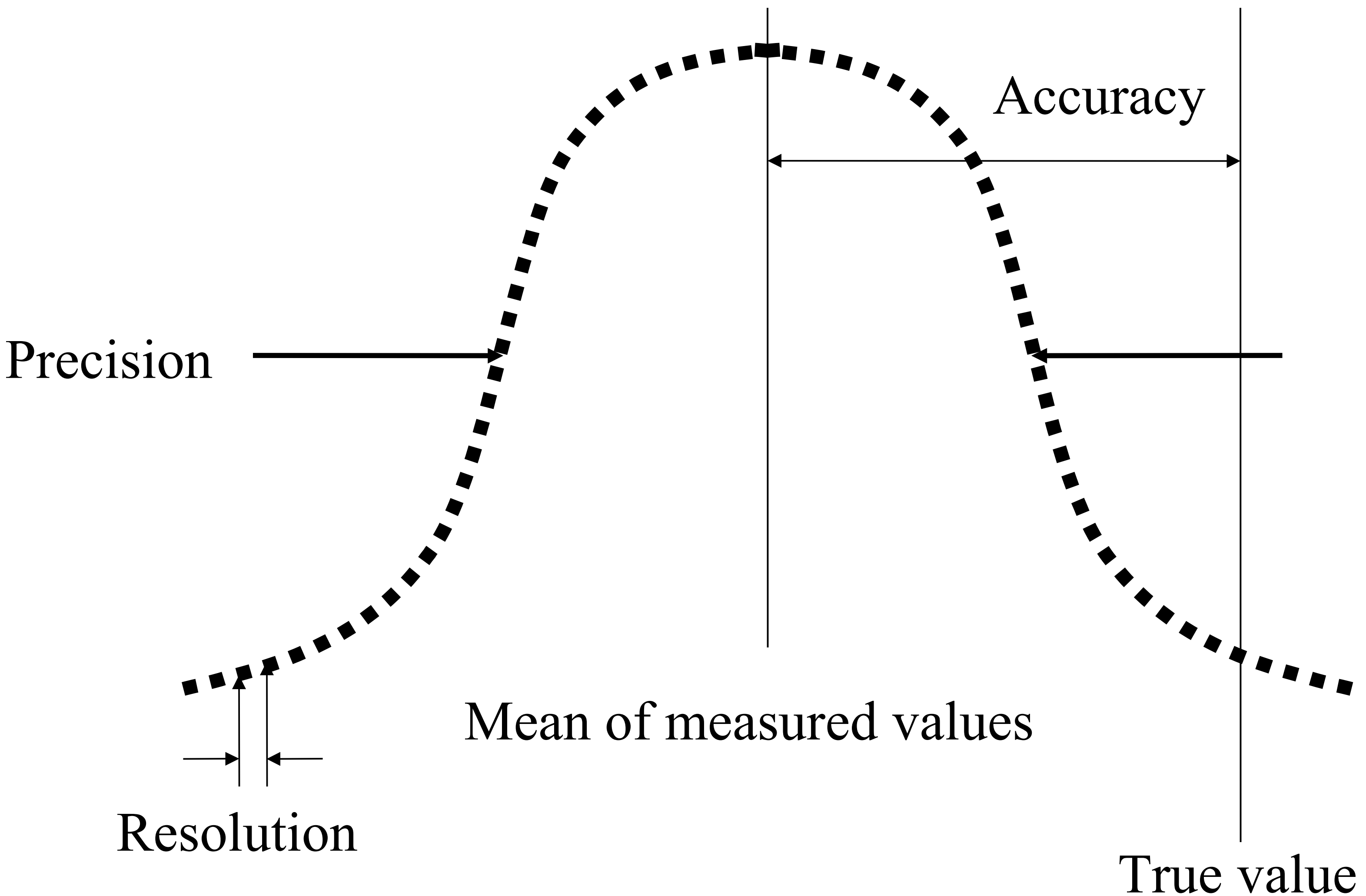
Accuracy, precision, resolution I

- **Resolution** is the fineness to which an instrument can be read.
- **Precision** is the fineness to which an instrument can be read *repeatably and reliably*.
- **Accuracy** is correctness (i.e., how close to reality)

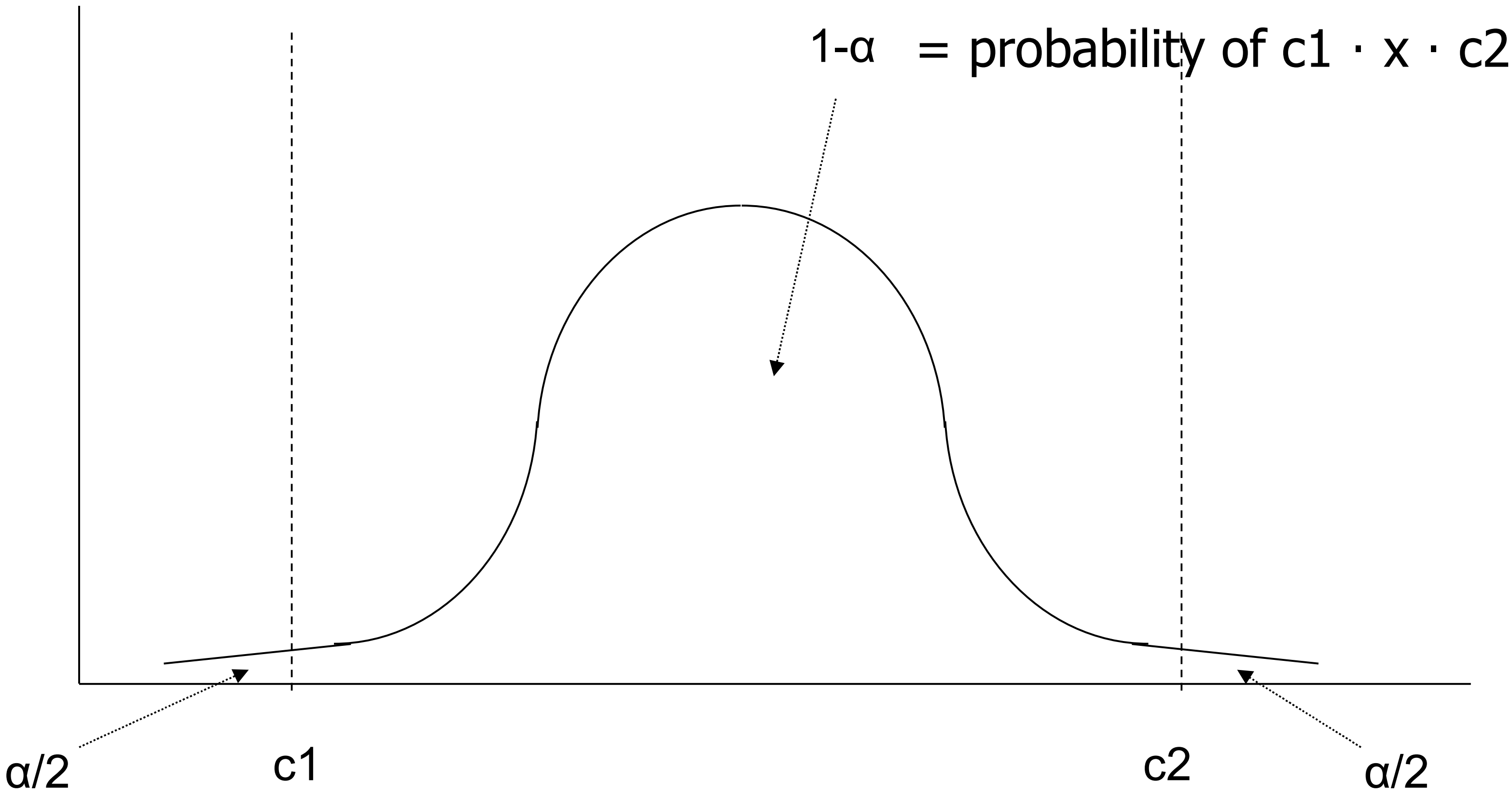
Accuracy, precision, resolution II

- Systematic errors → accuracy
  - How close mean of measured values is to true value
  - Hard to determine true accuracy
  - Relative to a predefined standard
    - E.g. definition of a "second"
- Random errors → precision
  - Repeatability of measurements
  - Dependent on tools
- Characteristics of tools → resolution
  - Smallest increment between measured values
  - Quantify amount of *imprecision* using statistical tools

Frequency of measuring specific value



Confidence interval for the mean



Normalize *x*

$$z = \frac{\bar{x} - x}{s / \sqrt{n}}$$

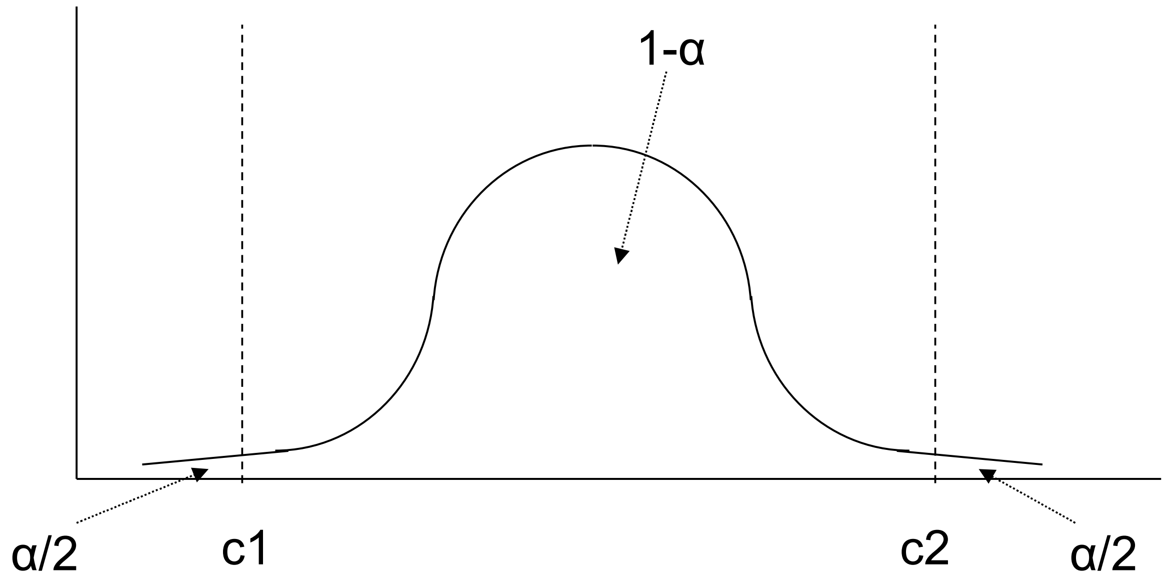
$$n = \text{number of measurements}$$

$$\bar{x} = \text{mean} = \sum_{i=1}^{n} x_i$$

$$s = \text{standard deviation} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}$$

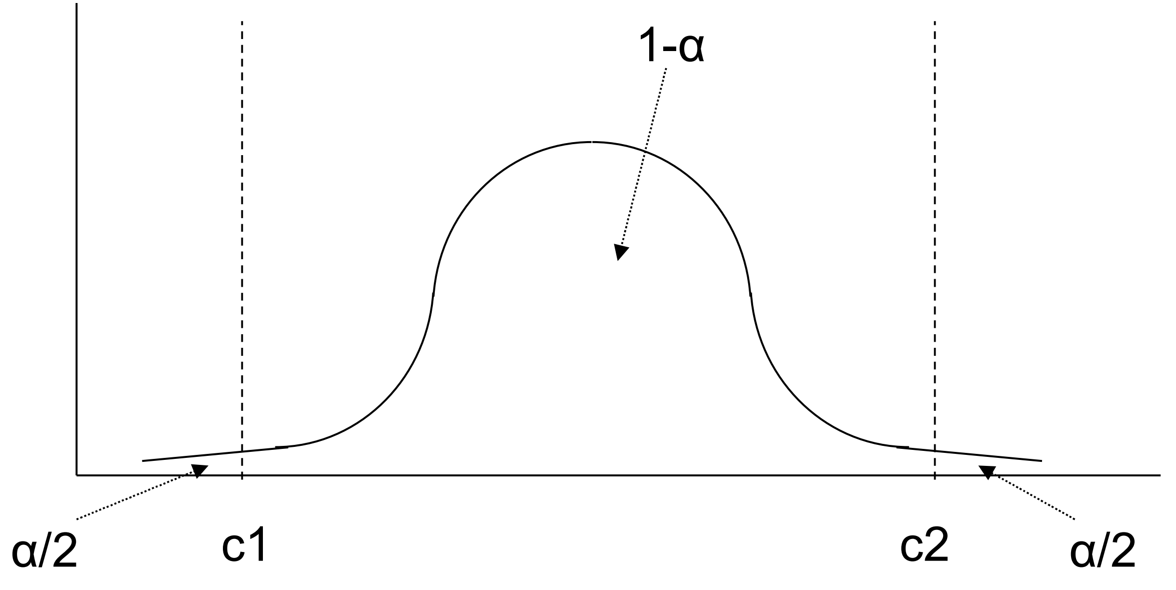
Confidence interval for the mean (2)

- Normalized z follows the Student's t distribution
  - $(n\text{-}1)$ degrees of freedom
  - Area left of $c_2 = 1 - \alpha/2$
  - Tabulated values for t



Confidence interval for the mean (2)

- As $n \to \infty$, normalized distribution becomes Gaussian (normal)



An example

Experiment	Measured value
1	8.0 s
2	7.0 s
3	5.0 s
4	9.0 s
5	9.5 s
6	11.3 s
7	5.2 s
8	8.5 s

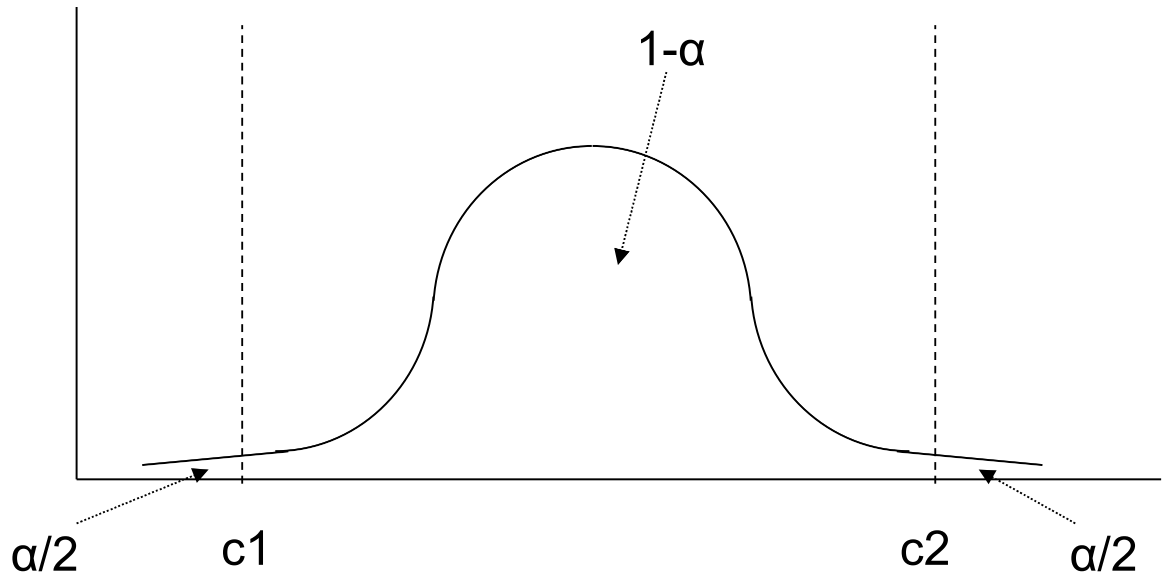
An example (2)

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = 7.94$$

$$s = \text{sample standard deviation} = 2.14$$

An example (3)

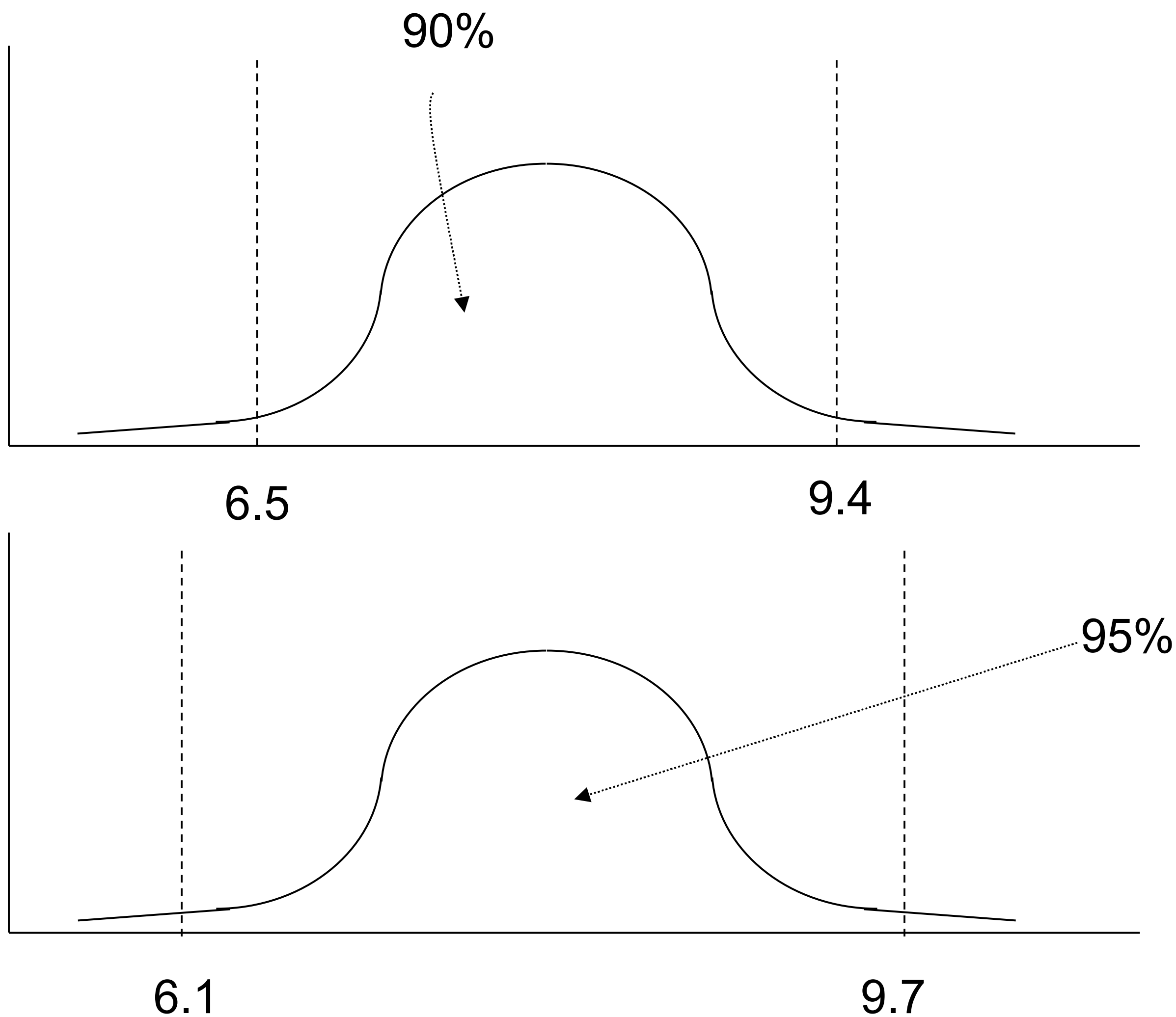
- 90% CI → 90% chance that the measured value is in the interval
- 90% CI → $\alpha = 0.10$



An example (4)

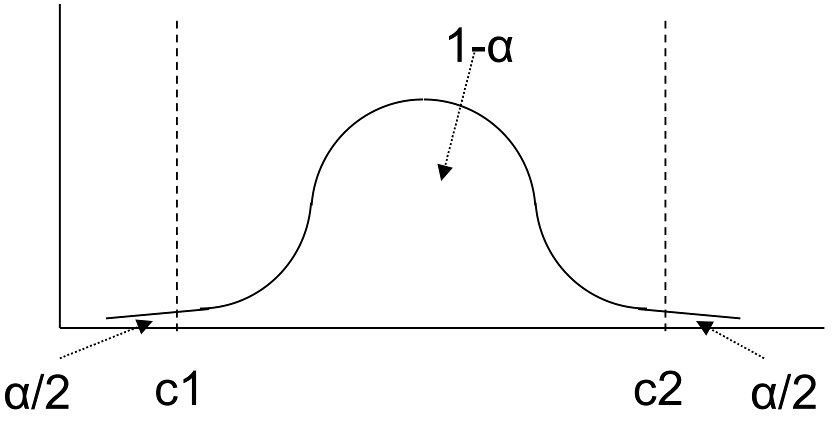
- 90% CI = [6.5, 9.4]
  - 90% chance value is between 6.5, 9.4
- 95% CI = [6.1, 9.7]
  - 95% chance value is between 6.1, 9.7
- Why is interval wider when we are more confident?

Higher confidence → Wider interval?



Key assumption

- Measurement errors are Normally distributed.
- Is this true for most measurements on real systems?



Key assumption (2)

- Saved by the Central Limit Theorem

  Sum of a "large number" of values from any distribution will be Normally (Gaussian) distributed.

- What is a "large number?"
  - Typically assumed to be $> \approx 6$ or 7
  - But in our case often millions or billions

How many measurements?

- Width of interval inversely proportional to $\sqrt{n}$
- Want to minimize number of measurements
- Find confidence interval for mean, such that:
  - $P(\text{actual mean in interval}) = (1 - \alpha)$

How many measurements (2)?

- But n depends on knowing mean and standard deviation!
- Estimate s with small number of measurements
- Use this s to find n needed for desired interval width