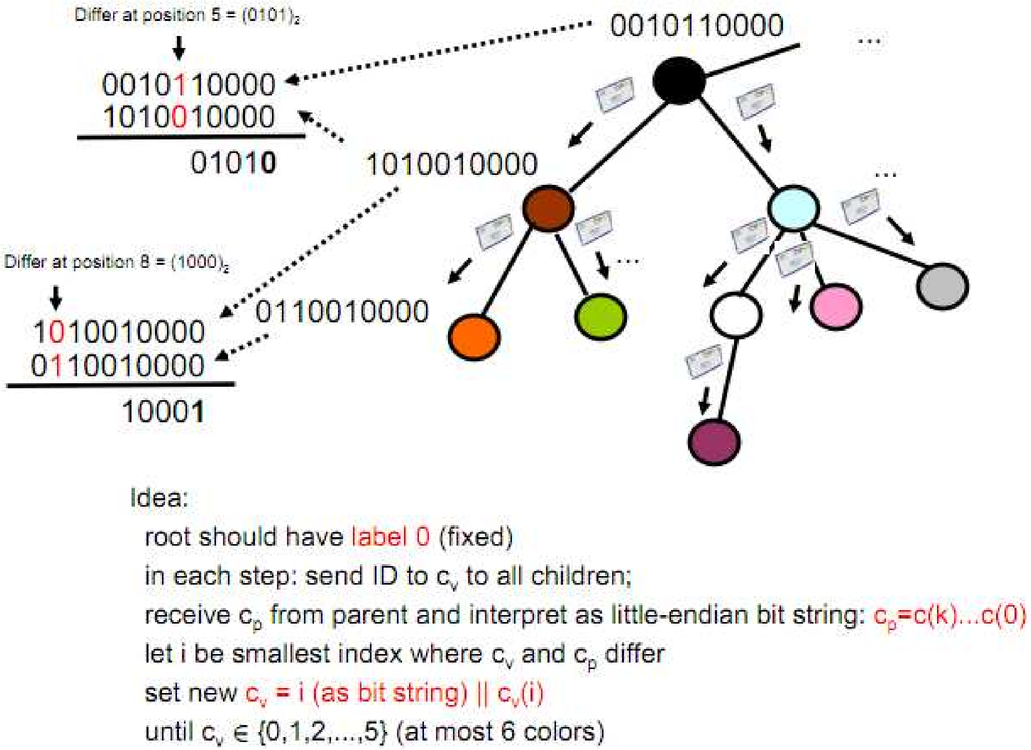
Foundations of Distributed Systems:

Locality Lower Bounds

Vertex Coloring: Results so far?

E.g., on trees in log*(n) time, down to 6 colors...



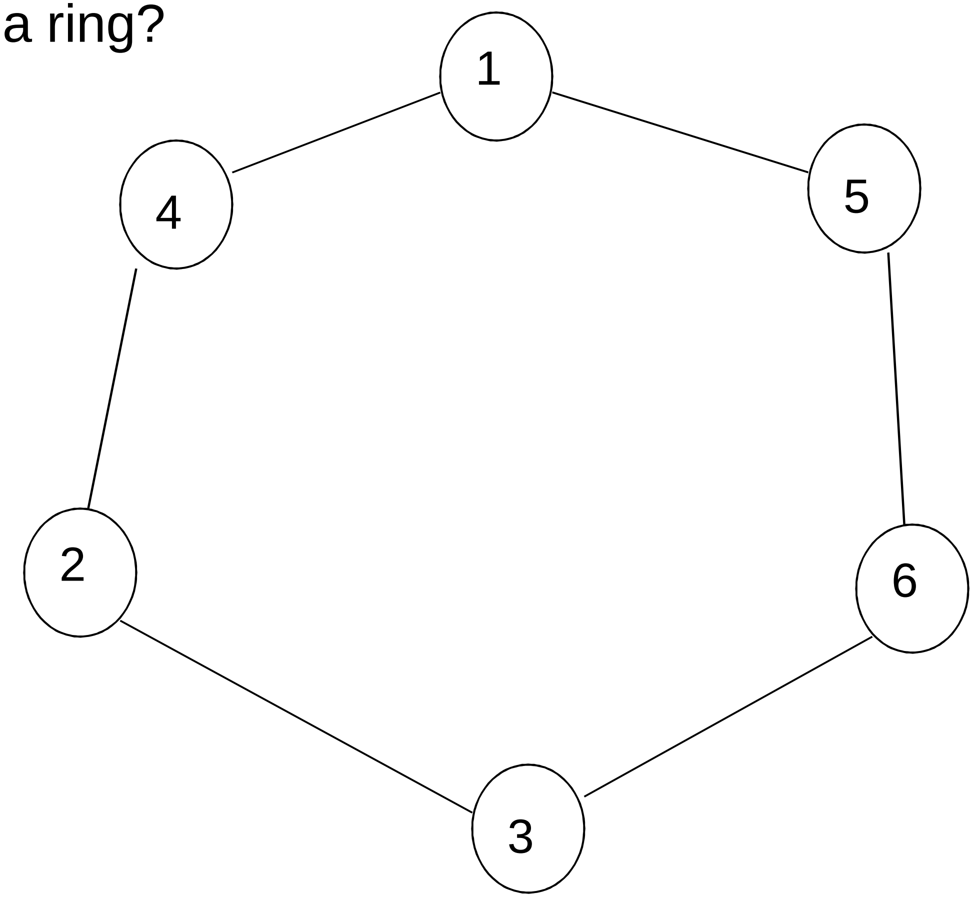
Round 1

Is this optimal??

... and then shift-down: down to 3 colors (same complexity).

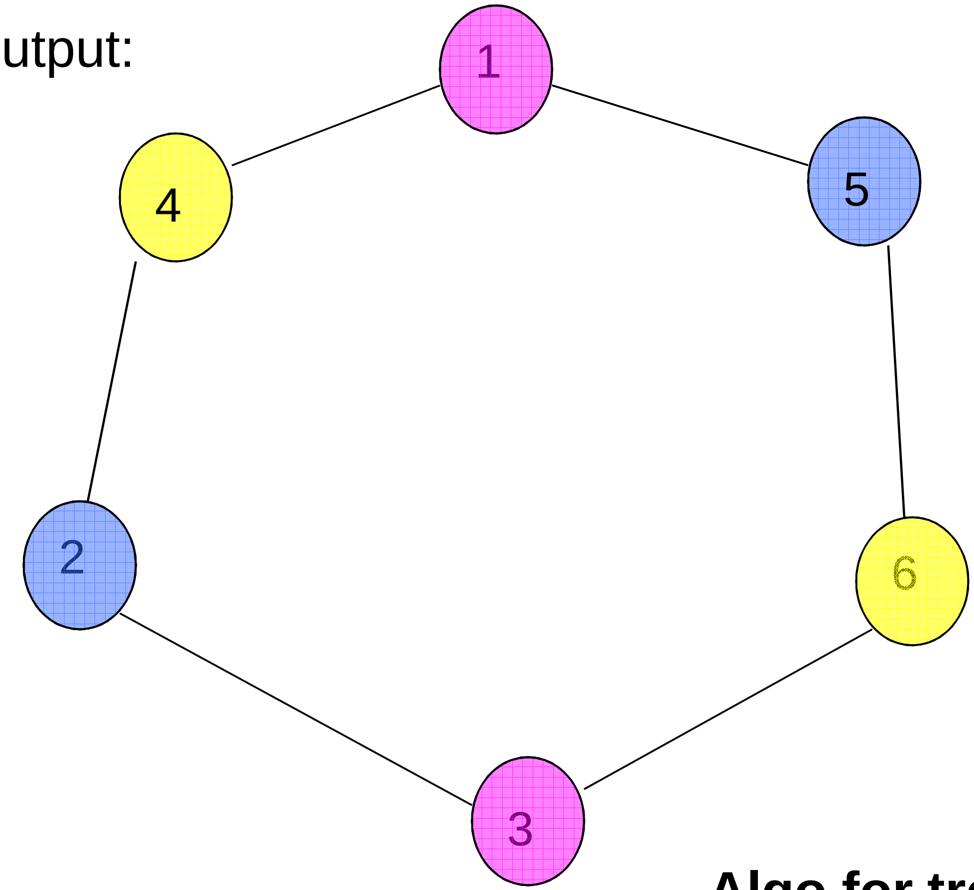
From trees to rings...

How to color a ring?



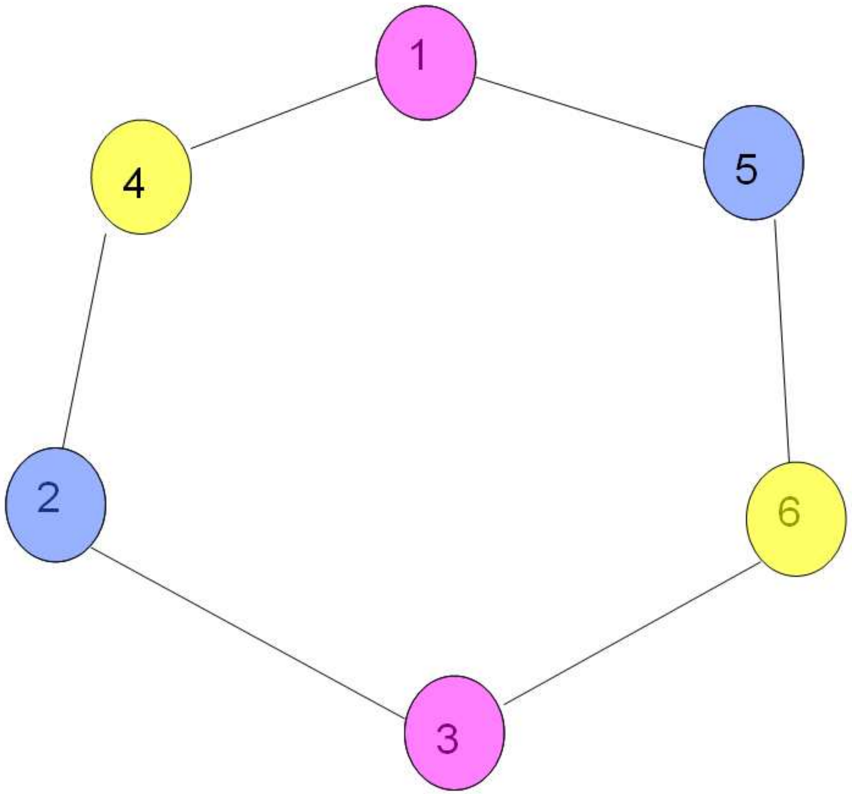
Ring Coloring

Possible output:



Algo for trees can be adapted!
[Exercise.]
So log*(n) time...!

Lower Bounds: First Thoughts and Outlook



Assume unique node IDs:

Lower bound for # colors without communication?

n

Lower bound for # colors with one communication round?

log n

Lower bound for # colors with two communication rounds?

log log n

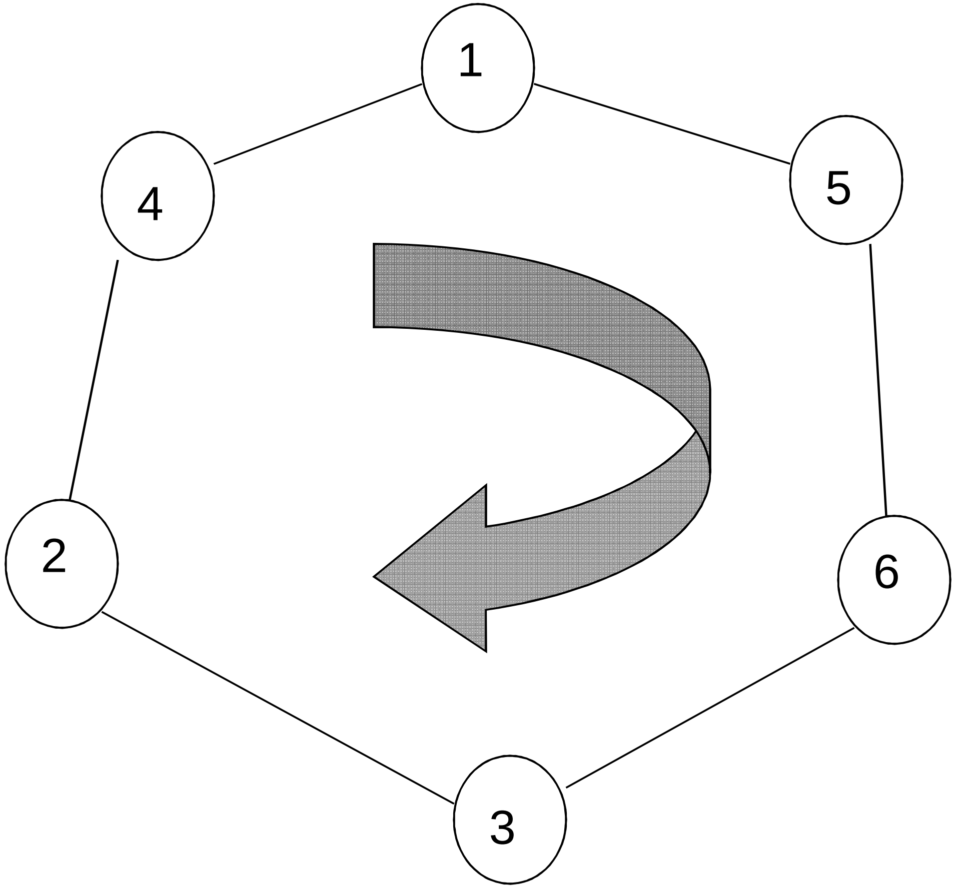
Lower bound for # colors with log* n communication rounds?

O(1)

Lower Bounds: For simple ring, not tree....

3-color a ring: log*(n) time is optimal!

How to prove?



Class of algos?
Need assumptions!

1. synchronous, **directed** ring
   (communication in both directions and nodes can differentiate between clockwise and counter-clockwise)
2. IDs from **1...n**
   (not in order, otherwise trivial!)
3. unbounded message size

The stronger assumptions for which the lower bound is still high the better for us!

Remember „local algorithm“
is symmetric: each node executes the same code! We will see:
dan differentiate only in terms of neighborhoods...

Canonical Form of Distributed Algorithm?

What can a distributed algorithm do or learn in r rounds?

1. Initially, all nodes only know their own ID
2. As information needs at least r rounds to travel r hops, a node can only learn about r-hop neighborhood!

Note that any local r-round algorithm can be brought into canonical form!

Canonical Form

1. First, in r rounds: send initial state to nodes at distance r
2. Then: compute output based on complete information about r-hop neighborhood

In other words: we can emulate any local algorithm by making all communication first and then do all local computations! Why?

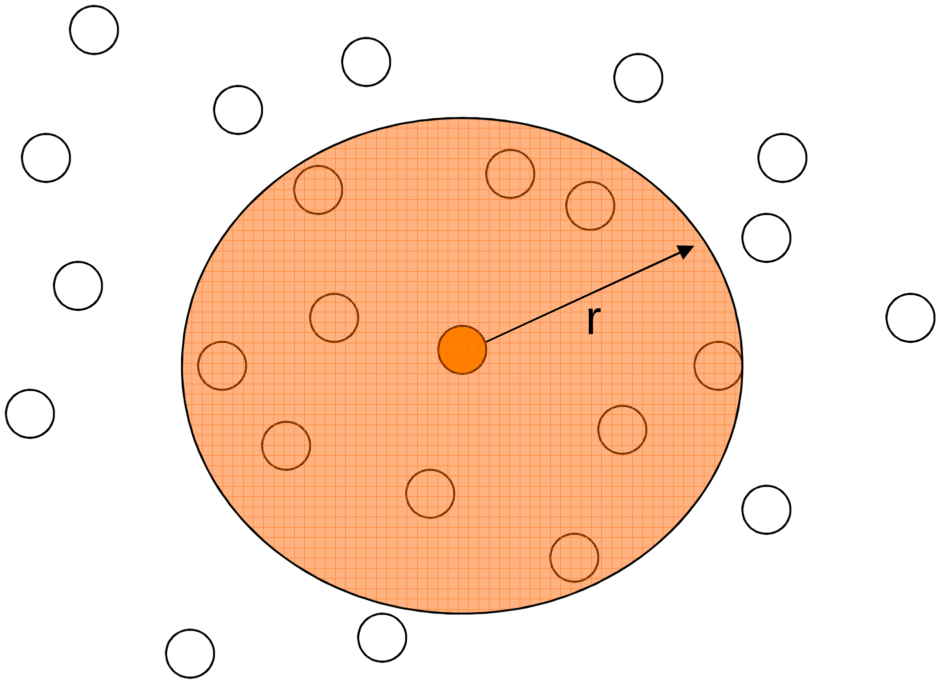
Example „leader election“:

Whether nodes only forward highest ID so far or whether all information is collected first and later selected does not make a difference!

No Deterministic Local Algorithms Can Do More...

We can do all communication first and then do all local computations!

How to prove this?

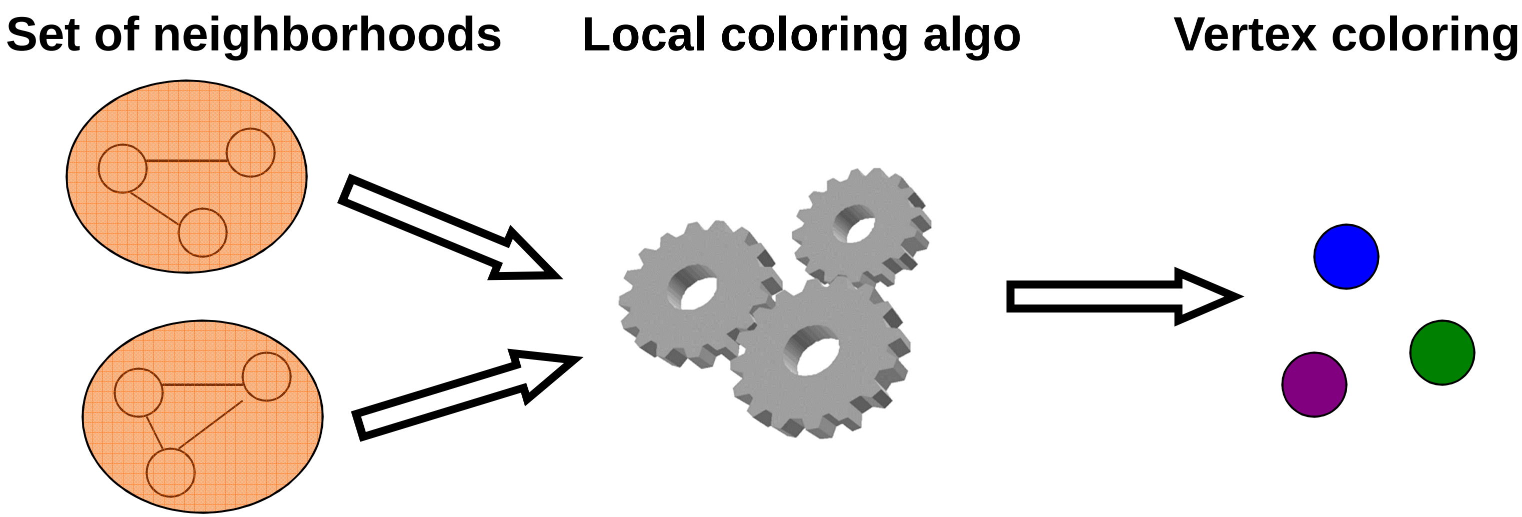


Let A be any r-round algorithm.
We can show that the canonical form algorithm C can compute all possible messages that A may send as well.
By induction over distance of nodes...:
if we can compute messages of first i rounds in (r-i+1)-neighborhood, we have all information to compute first (i+1) round messages in (r-i)-neighborhood.
So first trivial: Can compute all first messages in r-neighborhood.
Then: Can compute all second messages in next round. (But don't know what arrived externally...) Etc. See „Skript". ☺

Implication?

Takeaway

A local coloring algorithm can be seen as a function which takes neighborhoods and outputs colors.



Local Views

This motivates the following definition:

r-Hop View

We call the collection of the initial states of all nodes in the r-neighborhood of a node v the „r-hop view of v".

Due to our canonical form lemma, this means that:

Deterministic r-Round Algo

A deterministic r-round algorithm A is a function that maps every possible r-hop view to the set of possible outputs.

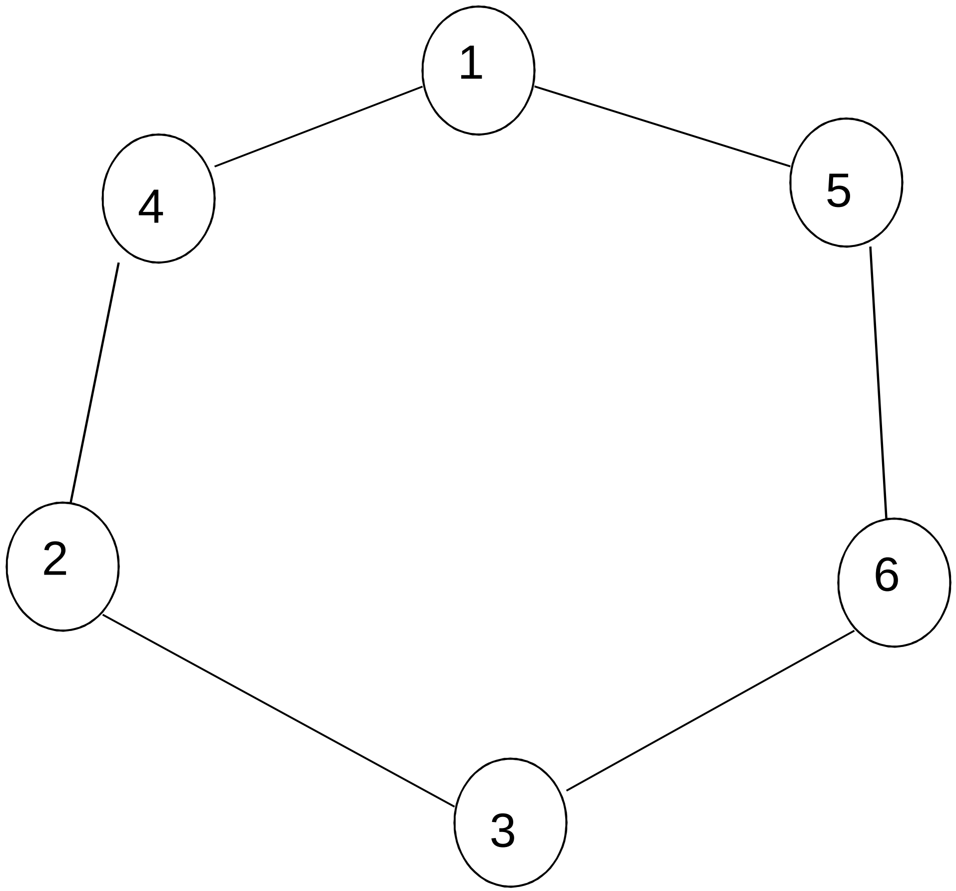
Implication for nodes with same view?
Must produce same output, in any algorithm!

So if any local algorithm can be emulated
by a canonic algorithm, the question remains:

How good can a canonic algorithm maximally be?

Rings

How do r-hop views of our rings look like?
E.g., 1-hop view of 4?



Rings

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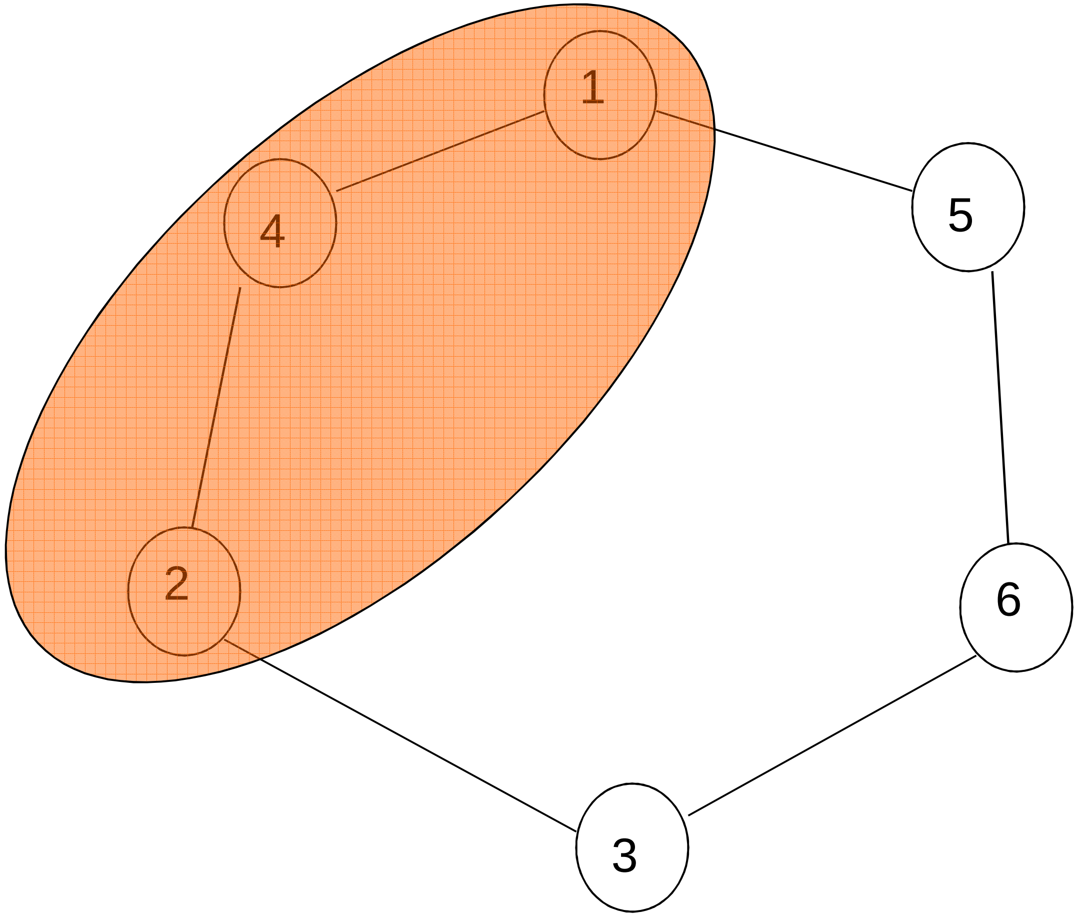
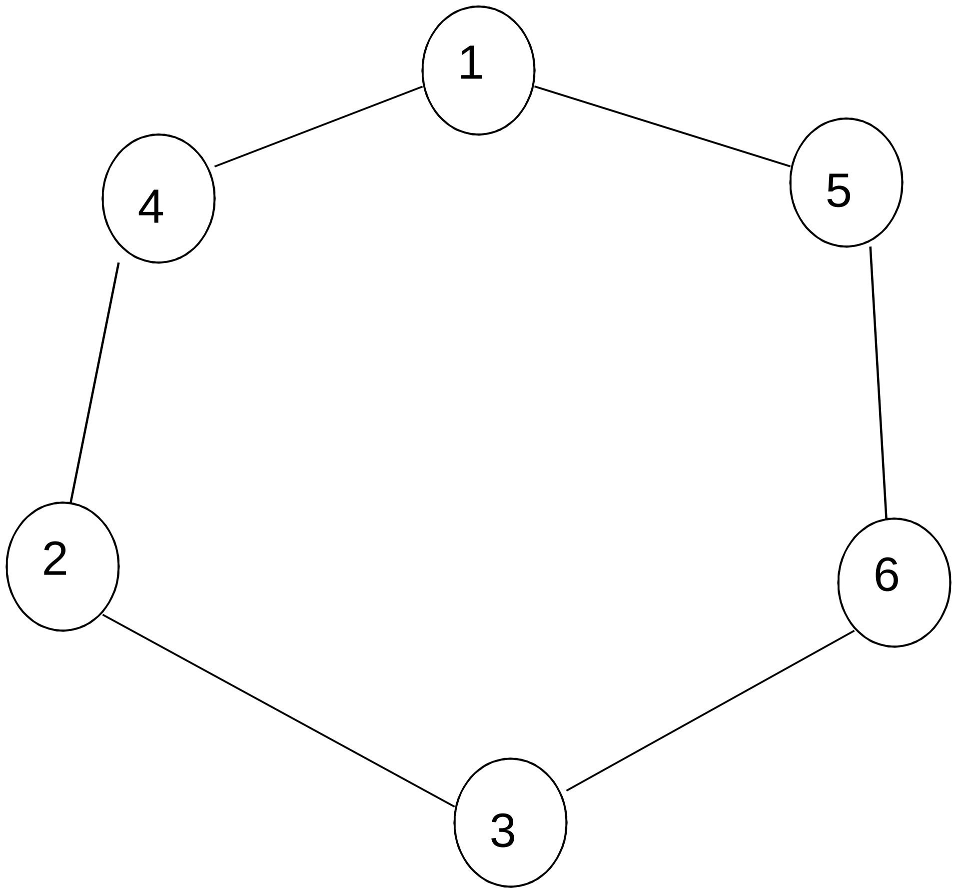


Rings

How do r-hop views of our rings look like?
E.g., 2-hop view of 4?



Rings

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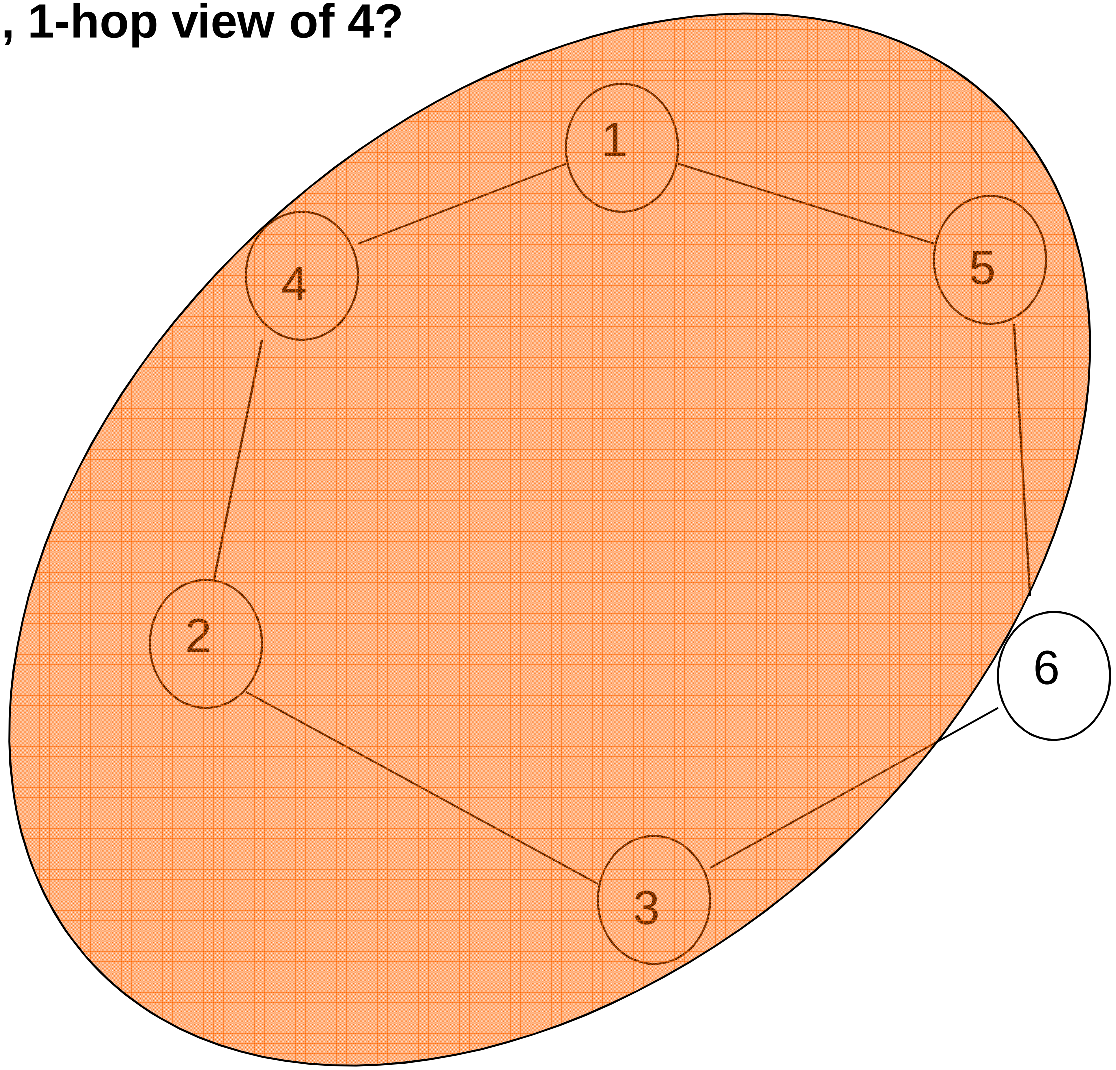


Ring Colorings

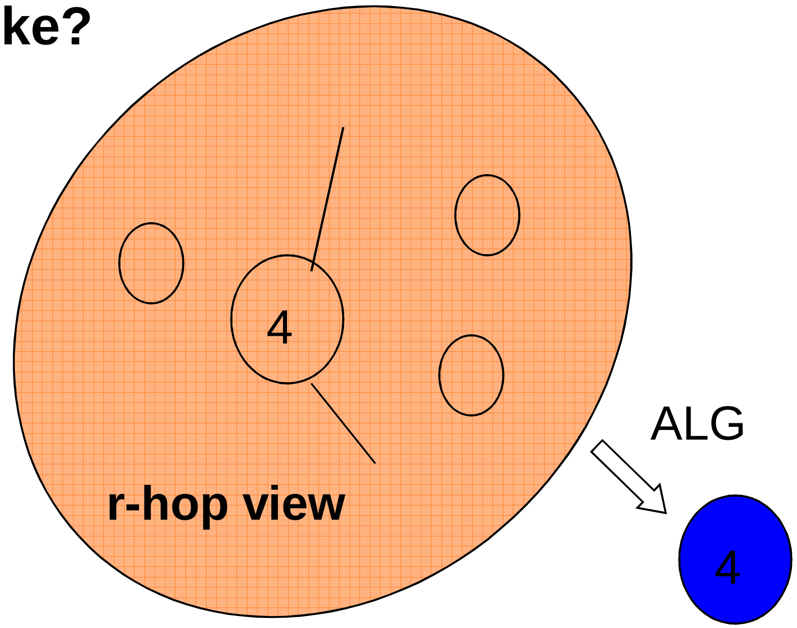
How do r-hop views of our rings look like?

Generally:

The r-hop view of a ring is a (2r+1) tuple:

$$(\mathbf{I_{-r}, I_{-r+1}, \ldots, I_0, \ldots, I_r})$$

where I_0 is ID/label of considered node v.



A deterministic coloring algorithm maps these tuples to colors!

Question: why tuple and not set? Sense of orientation! ☺

Ring Colorings

When is a coloring valid?

Consider two r-hop views:

$(l_{-r}, l_{-r+1}, \ldots, l_0, \ldots, l_r)$

and

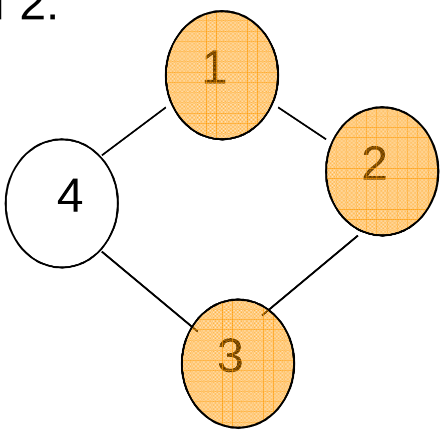
$(l'_{-r}, l'_{-r+1}, \ldots, l'_0, \ldots, l'_r)$

where $l'_i = l_{i+1}$ for $-r \leq i \leq r-1$ and $l'_r \neq l_i$ for $-r \leq i \leq r$, so what?

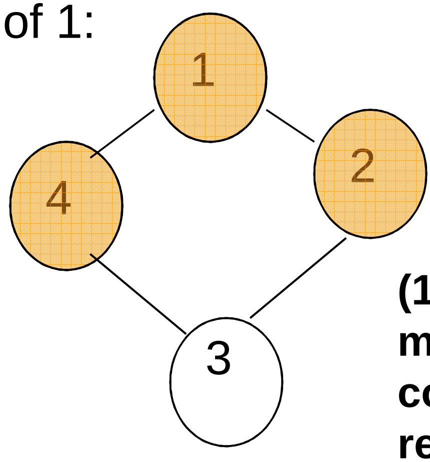
Then the two views can originate from adjacent nodes in the ring! So?

So every algorithm needs to assign different colors to the two views!

1-hop view of 2:



1-hop view of 1:



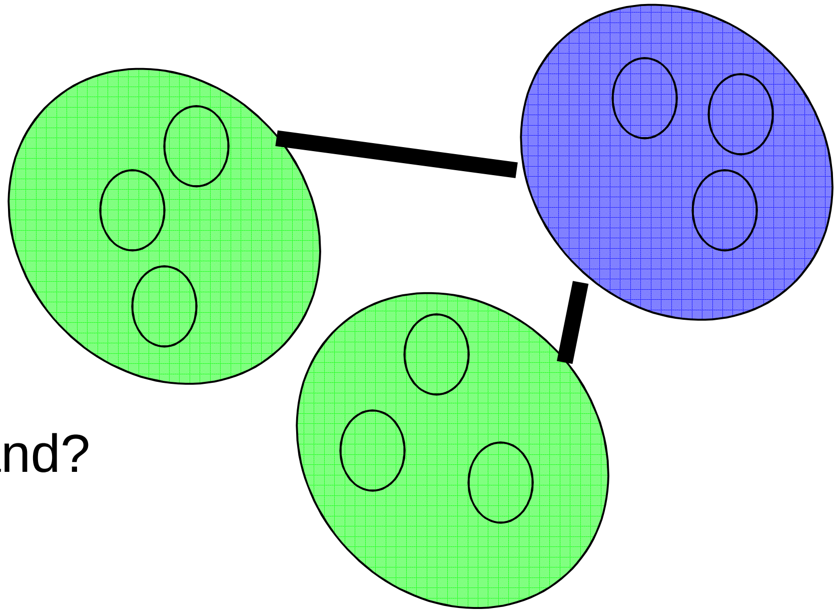
(1,2,3) and (4,1,2) must give different colors (for 1 and 2, respectively!)

Neighborhood Graphs?

What if we define a neighborhood graph: neighborhoods are nodes, and connected if they are conflicting (i.e., views may originate from two adjacent nodes)?

Assume we color the neighborhood graph as follows: „view node“ has color of the node the neighborhood is computed from by deterministic local r-round algo.

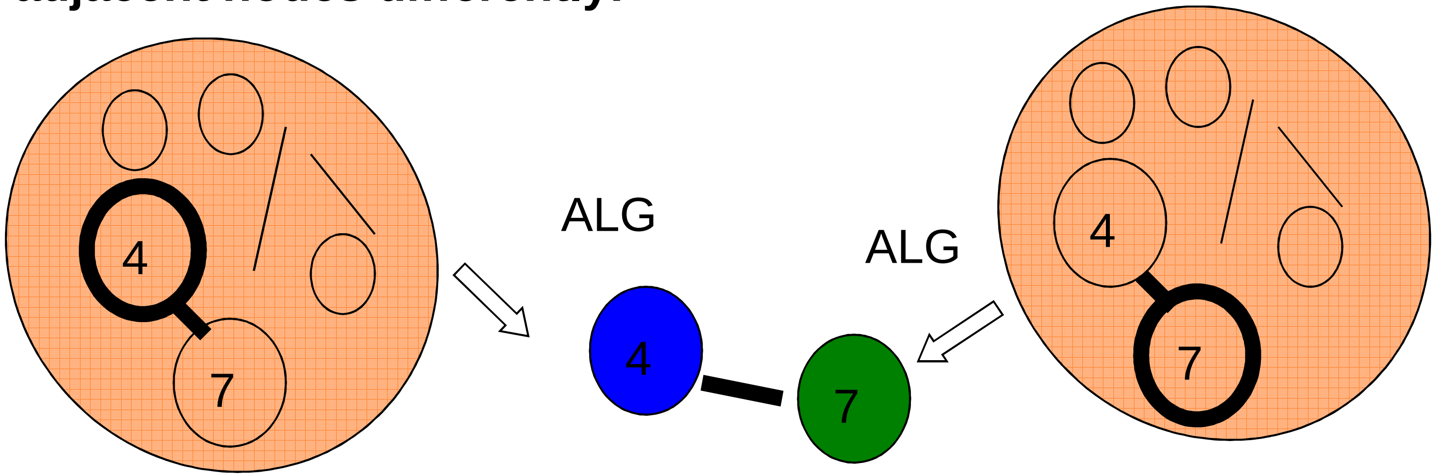
How does the coloring of the neighborhood graph look like then?



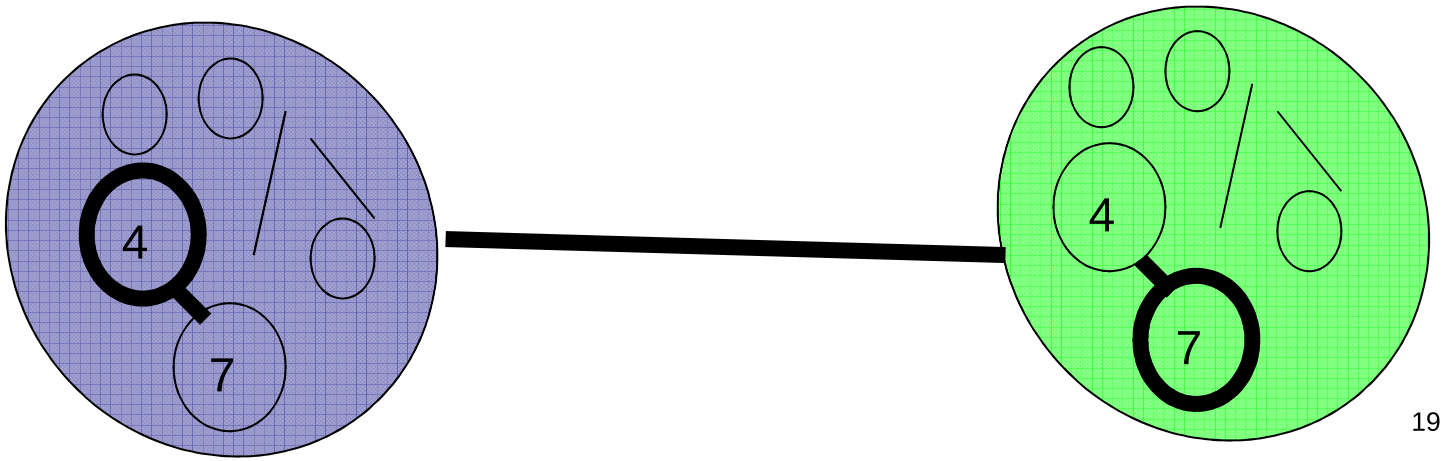
Same neighborhood = same color, and?

Neighborhood Graphs?

Given collected neighborhoods, canonic coloring ALG colors adjacent nodes differently:



So corresponding views/nodes in neighorhood graph must have different colors too, so valid coloring for neighborhood graph:



Neighborhood Graph

„Formal“ definition:

Neighborhood Graph

The **r-neighborhood graph $N_r(G)$** consists of all r-hop views of G (for all nodes) which are connected iff they could originate from two adjacent nodes.

This lemma motivates the concept:

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\chi(N_r(G)) \leq c$.

Proof?

Neighborhood Graph

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\boldsymbol{\chi(N_r(G)) \leq c}$.

Proof:

Because r-round algorithm defines legal coloring on neighbhorhood graph!
(Everything else could yield conflict: neighborhood graph contains all possible conflicts.)

We know: local coloring algo is a function that maps r-hop view to color, so to every node of $N_r(G)$...
This coloring is legal: by the definition of r-hop neighborhood graphs, adjacent nodes of $N_r(G)$ must have different colors, since the corresponding nodes in the underlying graph are also adjacent.
(But maybe slightly more than c colors are needed, so "$\leq$"...)

QED

So how do neighborhood graphs of rings look like? How to color them? And how to exploit the lemma to get a lower bound?

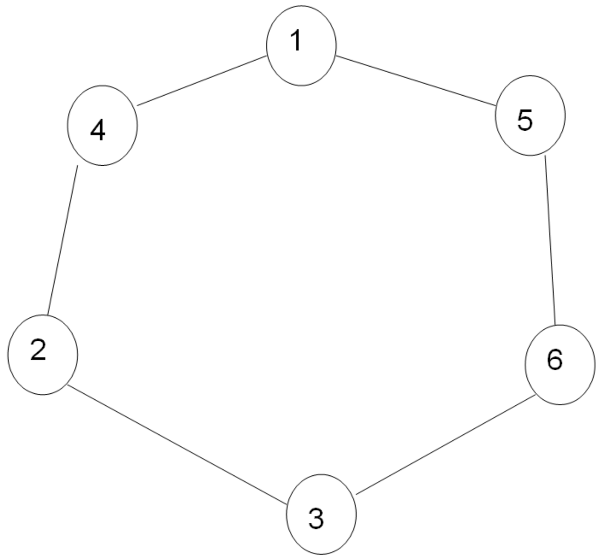
Roadmap

How to find a good lower bound with this lemma?

We have to show that $\chi(N_r(G))$ is small only for a large r...

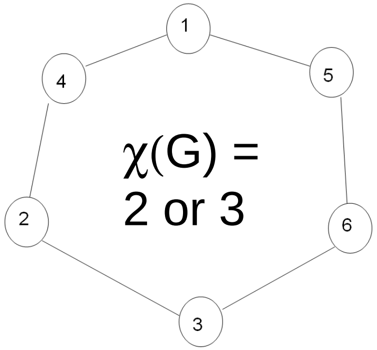
So how does $N_r(G)$ of a ring look like?

For example of our initial ring graph?

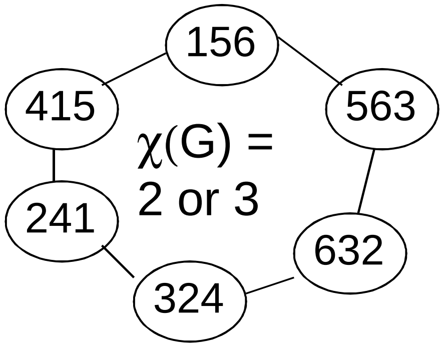


N_r(Given Ring)?

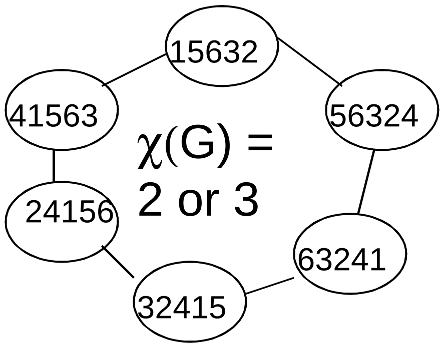
0-hop neighborhood graph?



1-hop neighborhood graph?



2-hop neighborhood graph?



So 0 or 1 round to 3-color?!?

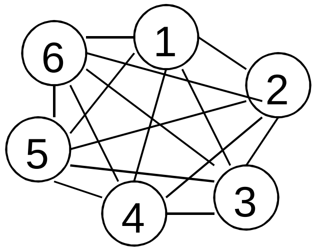
Attention: We are interested in neighborhood graphs of families of graphs / rings!

A given graph is easy (neighborhoods trivial)! ☺

N_r(Ring)?

r-hop neighborhood graph for ring family (n=6 known)?

$$\mathbf{N_0 = ?}$$

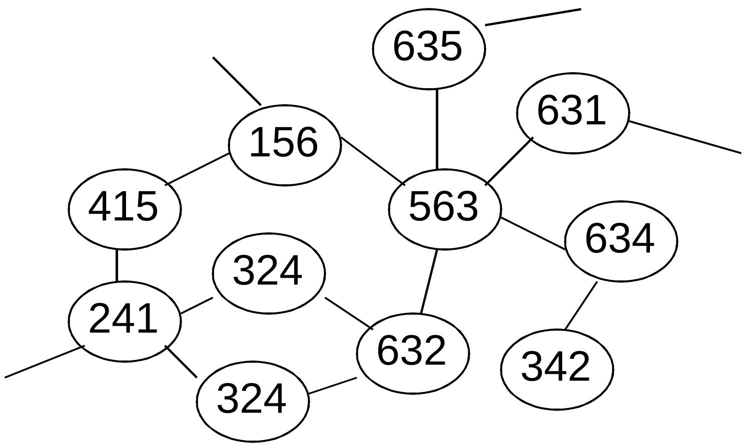


Complete graph: every node could be neighbor of every other node

$$\chi(\mathbf{N_0}) = ? \quad = n$$

Any 0-local algorithm can only choose its ID as a color...: n colors

$$\mathbf{N_1 = ?}$$



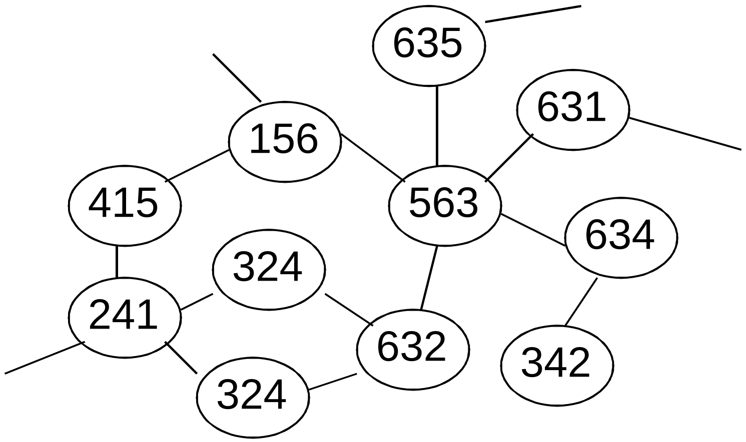
$$\chi(\mathbf{N_1}) = ???$$

Not easy although quite regular...

N_r(Ring)?

What happens for larger neighborhoods?

Intuitively, the larger the considered neighborhood, the less conflicts are possible! Chromoatic number declines for larger r... (We will see: in logarithmically „per hop“!)
At some point, the graph family member is clear!

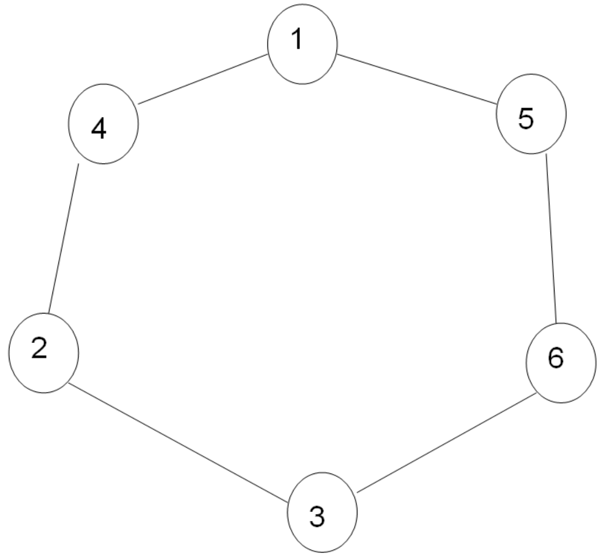


Main question now: What is $\chi(N_r(\text{Ring}))$??

Difficult... So let‘s focus on a graph which is similar, but has less conflicts and hence its chromatic number can be used instead for the lower bound!

What graphs are good then?

E.g., subgraphs...: less conflicts, so weaker lower bound when applying our lemma!



Overview of Proof

Lower Bound

Any deterministic distributed algorithm to color a ring with 3 or less colors needs at least $(\log^* n)/2-1$ rounds.

Deterministic r-Round Algo

A deterministic r-round algorithm A is a function that maps every possible r-hop view to the set of possible outputs.

1. **Canonic k-hop local algorithm with views**

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\chi(N_r(G)) \leq c$.

2. **Chromatic number of NG is lower bound for k-hop local algo.**

Lemma

Viewed as an undirected graph, $B_{2r+1,n}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,…,n}.

3. **Helper graph with no larger number of colors**

Lemma

$B_{k+1,n} = DL(B_{k,n})$

Lemma

$\chi(DL(G)) \geq \log_2(\chi(G))$

4. **Recursive construction and lower bound on colors.**

Lemma

$\chi(B_{1,n}) = n$ **and** $\chi(B_{k,n}) \geq \log^{(k-1)} n$

5. **Apply it to ring!**

Neighborhood Graph of Ring

Instead of defining neighborhood graphs for rings:

$B_{k,n}$ Graph

Assume two integers k,n where $n \geq k$. The $\mathbf{B}_{k,n}$ graph consists of the nodes of k-tuples of increasing node labels (from {1,...,n}). There is a directed edge from node α to node β iff $\forall$ i $\in$ {1,...,k-1}: $\beta_i = \alpha_{i+1}$.

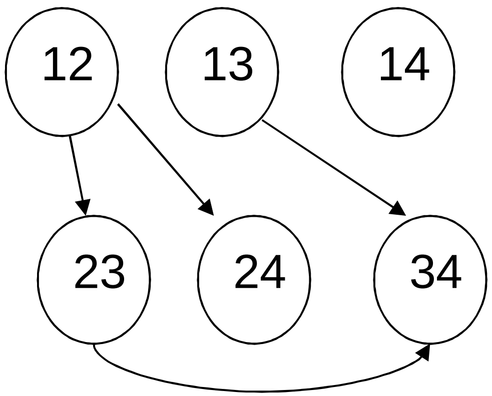
Example: k=2, n=4

$V(B_{k,n})$ = ?

= {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

$E(B_{k,n})$ = ?

= {((1,**2**),(**2**,3)), ((2,**3**),(**3**,4)),
((1,**2**),(**2**,4)) ((1,**3**),(**3**,4))}



Neighborhood Graph of Ring

What does this have to do with rings?!

Lemma

Viewed as an undirected graph, $\mathbf{B_{2r+1,n}}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,...,n}.

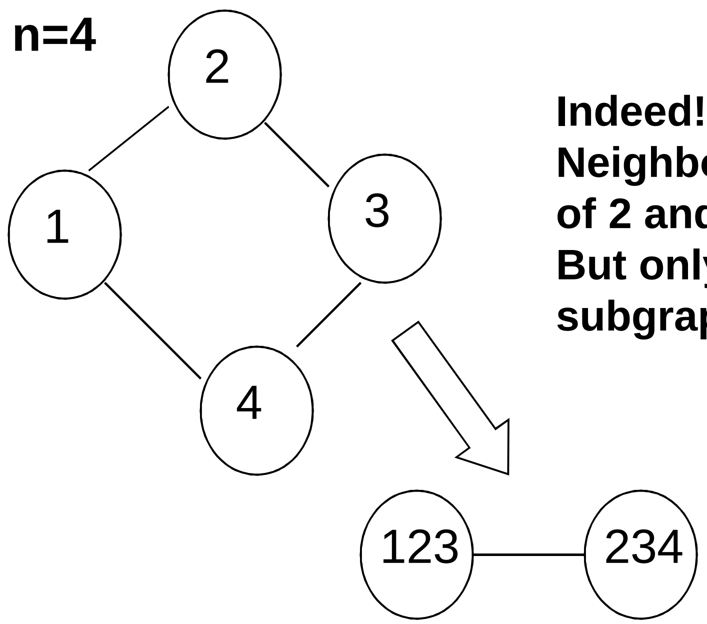
Example: Neighborhood r=1 (so k=3), n=4

$V(B_{k,n}) = ?$

$= \{(1,2,3),(1,2,4),$
$(1,3,4),(2,3,4)\}$

$E(B_{k,n}) = ?$

$= \{\{(1,\mathbf{2},3),(\mathbf{2},3,4)\}\}$



Indeed! Neighborhood of 2 and 3! But only a subgraph! (Why?)

Neighborhood Graph of Ring

Lemma

Viewed as an undirected graph, $\mathbf{B_{2r+1,n}}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,...,n}.

Proof?

The set of k-tuples of increasing labels is a subset of all the k-tuples / nodes (in our example, views of node 1 and 4 are missing).

Two nodes are only connected in $B_{2r+1,n}$ if there is also an edge in the neighborhood graph (because labels are ordered, the views must come from adjacent nodes): not more edges/conflicts.

QED

What does it mean?!

Chromatic number of $B_{2r+1,n}$ good for lower bound of our problem!

- We have to compute lower bound for $\chi(B_{2r+1,n})$!
- How? With another helper graph... ☺

Helper Graph

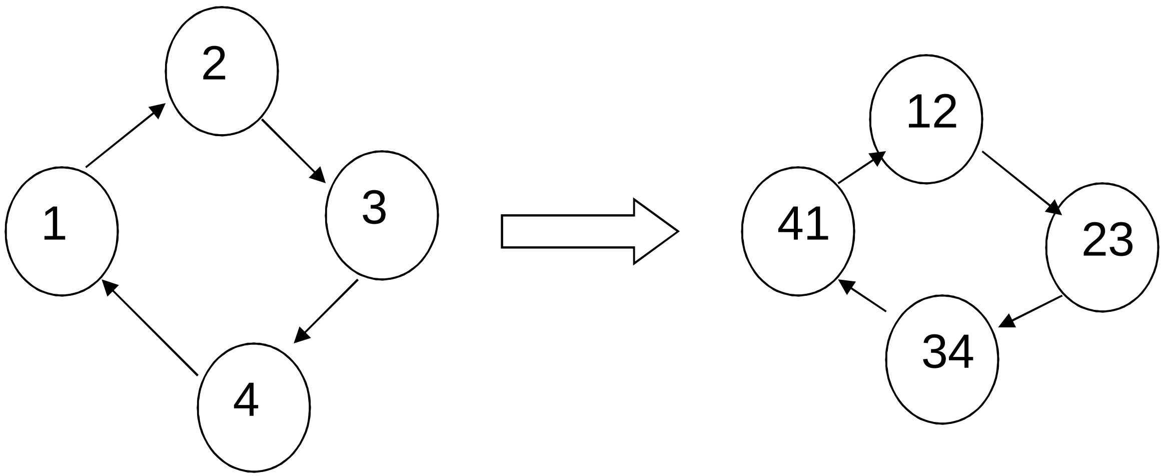
The following graph is helpful to analyze $B_{2r+1,n}$: What does it mean?

Diline Graph

The directed line graph (diline graph) **DL(G)** of a directed graph G=(V,E) is defined as follows: V(DL(G))=E, and there is a directed edge ((w,x),(y,z)) iff x=y.

In other words: DL(G) consists of the node representing the edges of G, and two nodes are connected if the corresponding edges „follow“ after each other.

Example:



What is the relation to $B_{k,n}$?!

Recursive Construction

$B_{k,n}$ can be recursively defined by directed line graphs!

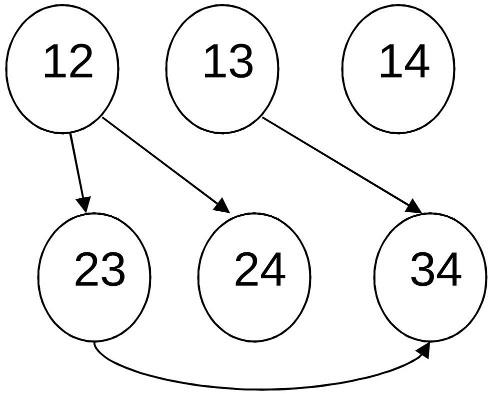
Lemma

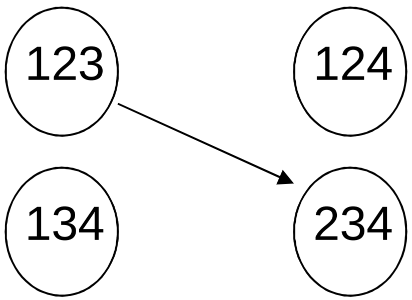
$B_{k+1,n} = DL(B_{k,n})$

Really? Example: k=2, n=4?

$V(B_{k,n}) = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

$E(B_{k,n}) = \{((1,\mathbf{2}),(\mathbf{2},3)),\ ((2,\mathbf{3}),(\mathbf{3},4)),\ ((1,\mathbf{2}),(\mathbf{2},4)),\ ((1,\mathbf{3}),(\mathbf{3},4))\}$





Example: k=3, n=4?

$V(B_{k,n}) = \{(1,2,3),(1,2,4),(1,3,4),(2,3,4)\}$

$E(B_{k,n}) = \{((1,\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{3},4))\}$

Recursive Construction

$B_{k,n}$ can be recursively defined by directed line graphs!

Lemma

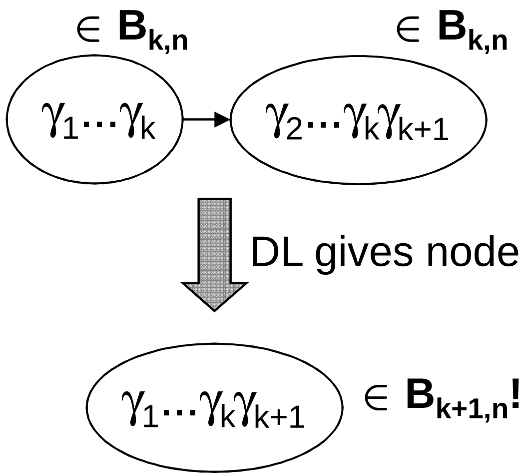
$B_{k+1,n} = DL(B_{k,n})$

Proof?

By the definition of $B_{k,n}$, two nodes α, β are connected if the first k-1 labels in β are the same as the last k-1 labels of α.

Therefore, the pair (α,β) can be represented by a (k+1) tuple $\gamma=(\gamma_1,\ldots,\gamma_{k+1})$ with $\gamma_1=\alpha_1$, $\gamma_i=\beta_{i-1}=\alpha_i$ for $2 \leq i \leq k$, and $\gamma_{k+1}=\beta_k$. The labels of γ are increasing too! So $B_{k+1,n}$ has the same node set as $DL(B_{k,n})$.

What about the edges?



Recursive Construction

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Lemma

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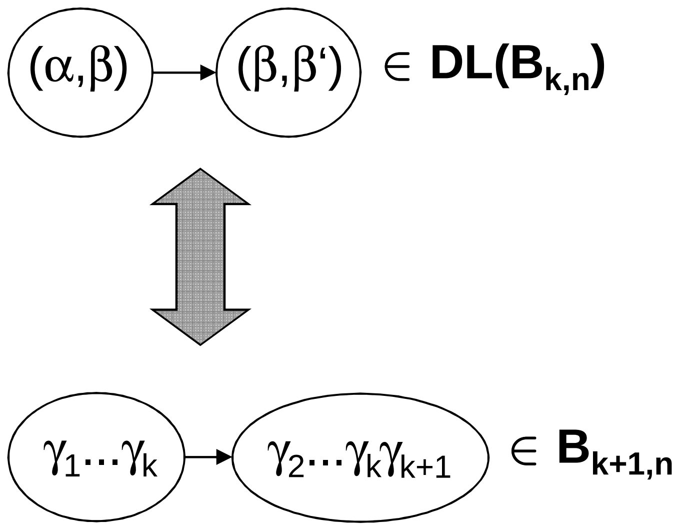


Proof (continued for edges...)

There is an edge between two nodes (α,β) and (α',β') of $DL(B_{k,n})$ if $\beta=\alpha'$.
This is equivalent to that the two corresponding $(k+1)$-tuples γ and γ' are neighbors in $B_{k+1,n}$:
the last k labels of γ are equivalent to the first k labels of γ'.

QED

So, $B_{k,n}$ graphs are simply „iterated line graphs"!

Chromatic Numbers

Holds for general graphs G!

Implication for colorings, coloring G vs DL(G)?

Lemma

$\chi(DL(G)) \geq \log_2(\chi(G))$

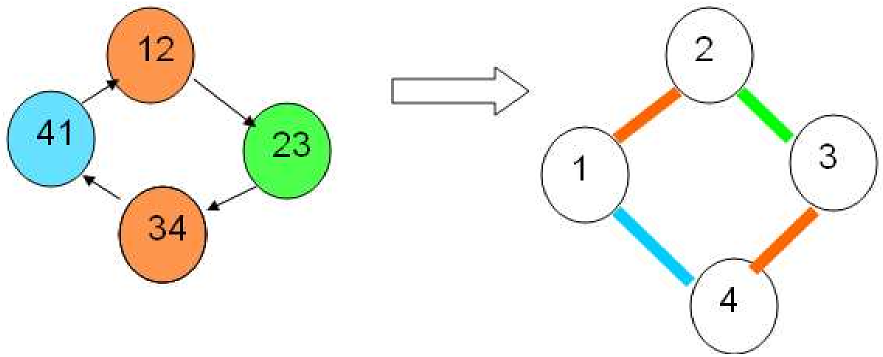
Proof idea?

Given a c-coloring of DL(G) we construct a **2^c coloring of G** (so minimal coloring of G can only be smaller).

How does coloring of G and DL(G) relate?

Note: A c-coloring of the diline graph DL(G) can be seen as a coloring of the edges of G such that no two adjacent edges have the same color (definition of DL(G)).

DL(G):



: G

Attention: these graphs are generally not rings of course! ☺

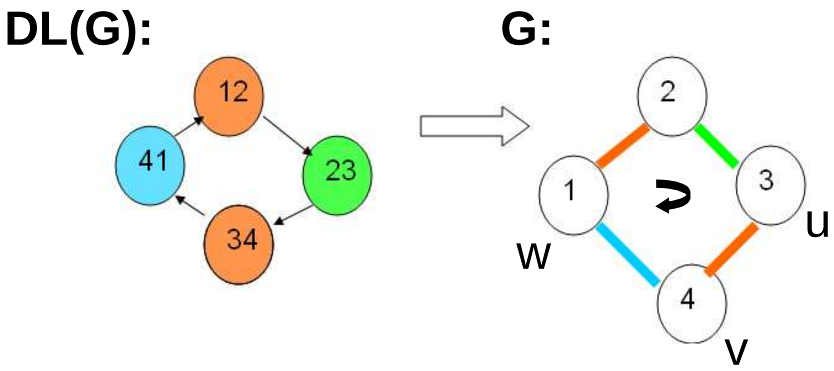
Implication for colorings, coloring G vs DL(G)?

Lemma

$\chi(DL(G)) \geq \log_2(\chi(G))$

Proof idea (continued...)

For a node $v \in G$, let $\mathbf{S_v}$ denote the set of colors of its outgoing edges in the graph. Let (u,v) be a directed edge in G and let x be the color of (u,v).
Thus: $x \in S_u$.



No edge (v,w) can have color x, so $x \notin S_v$, so $\mathbf{S_u \neq S_v}$: neighboring nodes in G must have different „out-edge-color-sets“!
We can use these color sets S to obtain a vertex coloring of G: the color of a node u is S_u. This coloring must be legal!
As we can have at most 2^c subsets (of c vertex colors of DL(G) and hence edge colors of G), the coloring has at most 2^c colors.

QED

Chromatic Numbers

Chromatic number of $B_{k,n}$?

Recall: Gives lower bound for r-hop coloring algo!
Intuitively: Each time the local view is increased, the chromatic number goes down at most by log!

Lemma

$$\chi(B_{1,n}) = n \text{ and } \chi(B_{k,n}) \geq \log^{(k-1)} n$$

Proof idea?

$B_{1,n}$ is the complete graph.
For larger k, it holds by induction due to our lemmas!

QED

Finally: Lower Bound

Combining everything gives our lower bound! ☺

Lower Bound

Any deterministic distributed algorithm to color a ring with 3 or less colors needs at least $(\log^* n)/2-1$ rounds.

Proof idea?

We need to show that $\chi(B_{2r+1,n})>3$ for all $r<(\log^* n)/2-1$.
We know that $\chi(B_{2r+1,n}) \geq \log^{(2r)} n$.
And $B_{2r+1,n}$ is subgraph of neighborhood graph we actually want!
The rest is simple maths...

QED

Summary of Proof

Lower Bound

Any deterministic distributed algorithm to color a ring with 3 or less colors needs at least $(\log^* n)/2-1$ rounds.

Deterministic r-Round Algo

A deterministic r-round algorithm A is a function that maps every possible r-hop view to the set of possible outputs.

1. **Canonic k-hop local algorithm with views**

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\chi(N_r(G)) \leq c$.

2. **Chromatic number of NG is lower bound for k-hop local algo.**

Lemma

Viewed as an undirected graph, $B_{2r+1,n}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,...,n}.

3. **Helper graph with no larger number of colors**

Lemma

$B_{k+1,n} = DL(B_{k,n})$

Lemma

$\chi(DL(G)) \geq \log_2(\chi(G))$

4. **Recursive construction and lower bound on colors.**

Lemma

$\chi(B_{1,n}) = n$ and $\chi(B_{k,n}) \geq \log^{(k-1)} n$

5. **Apply it to ring!**

Literature for further reading:

- Peleg‘s book (as always ☺)

End of lecture