Foundations of Distributed Systems:

Leader Election

Motivation

Leader Election

Nodes in network agree on *exactly one* leader.
All other nodes are followers.

Reasons for electing a leader?

Reasons for *not* electing a leader?

Motivation

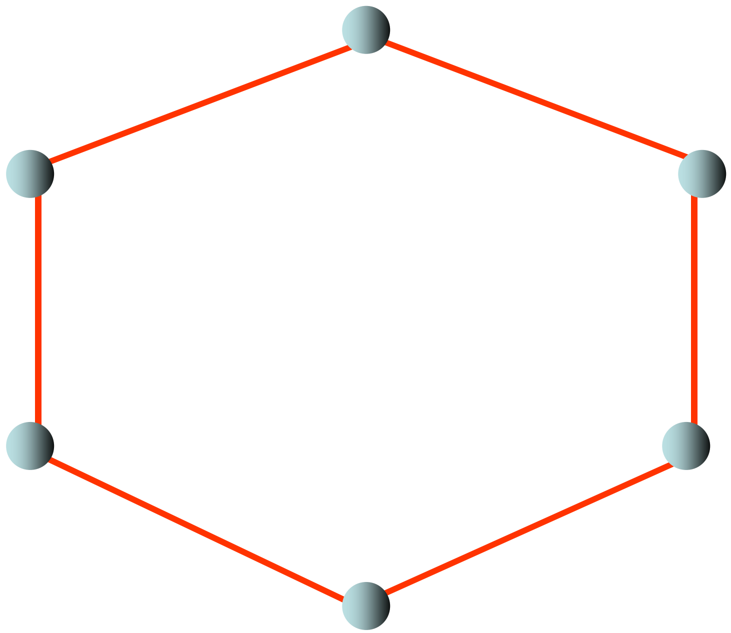
Reasons for electing a leader?

- Once elected, coordination tasks may become simpler
- For example: wireless medium access (break symmetry)

Reasons for *not* electing a leader?

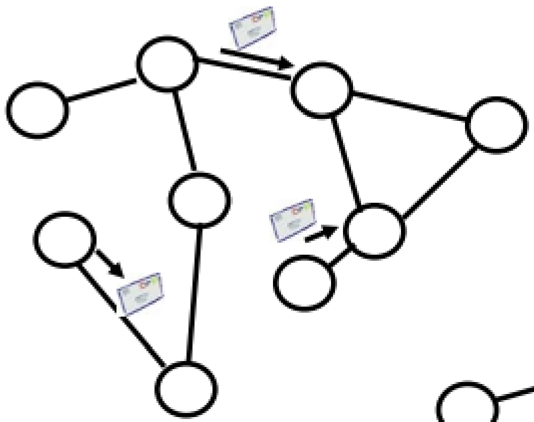
- Reduced parallelism?
- Self-stabilization needed: re-election when leader „dies“
- Leader bottleneck / single point of failure?

How to elect a leader in a ring?

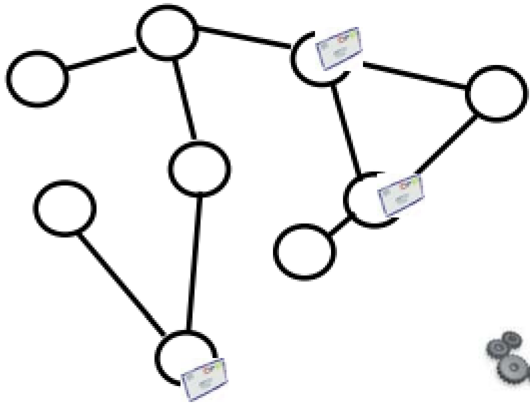


Model „Synchronous Local Algorithm“: Round

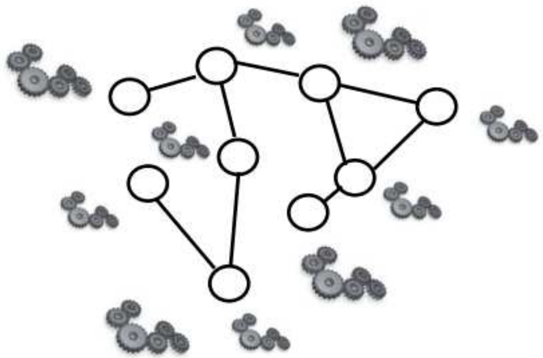
Send...



... receive...



... compute.



Anonymous System

Anonymous nodes do not have identifiers.

Theorem

In an anonymous ring, leader election is impossible!

Why?

Impossibility in Synchronous Ring

Theorem

In an anonymous ring, leader election is impossible!

First, note the following lemma:

Lemma

After round k of any deterministic algorithm on an anonymous ring, each node is in the same state s_k.

Proof idea?!

By induction: all nodes start in same state, and each round consists of sending, receiving and performing local computations. All nodes send the same messages, receive the same messages, and do the same computations. So they always stay in same state...

QED

So when a node decides to become a leader, then all others do too.

Discussion

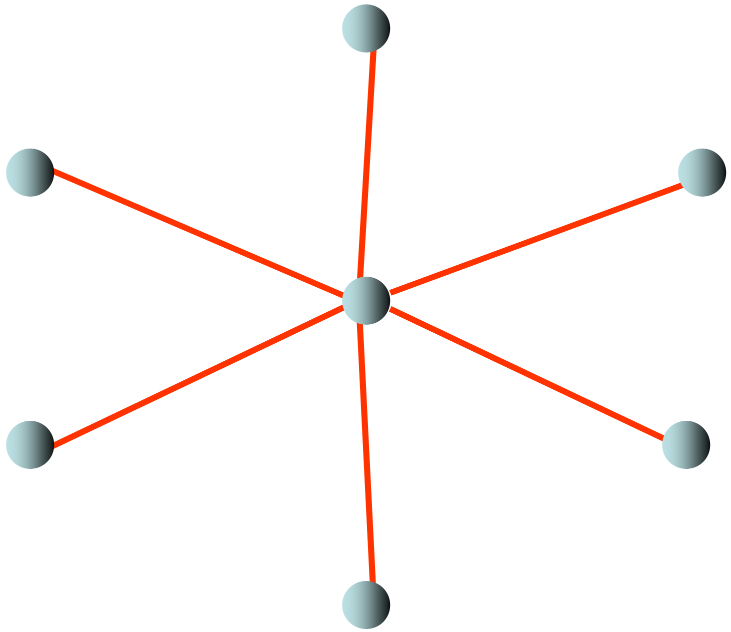
What is the basic problem?

Symmetry.... How could it be broken?

- How to elect a leader in a star?

- Randomization?

- What if nodes have IDs?



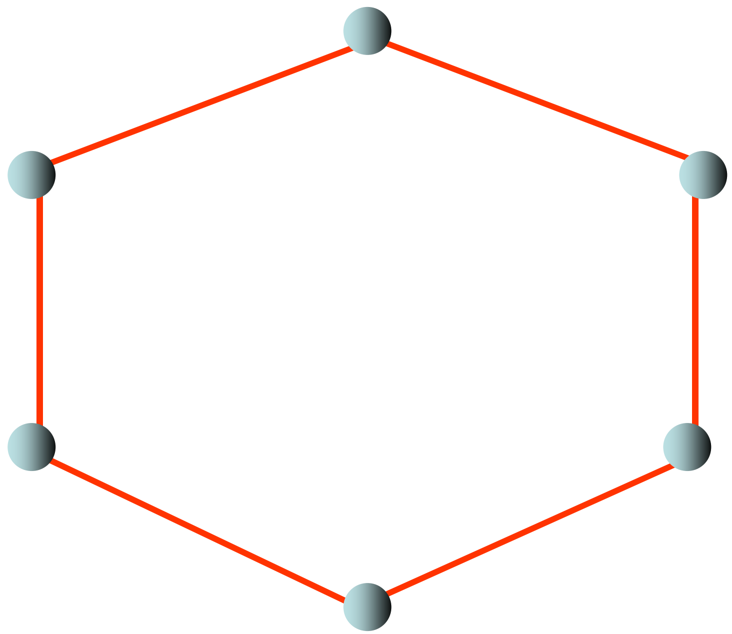
Asynchronous Ring

Let's assume:

- non-anonymous nodes with unique IDs
- asynchronous ring
- uniform ring: n unknown!
- no message losses etc.

Uniform System

Nodes do not know n.



How to elect a leader now?

Asynchronous Ring

Let‘s assume:

- non-anonymous nodes with unique IDs
- asynchronous ring

How to evaluate?
Criteria?
Asynchronous time?!

Algorithm Clockwise

each node v does the following:

- v sends a message with its ID v to clockwise neighbor (unless v already received a message with ID w>v)
- if v receives message w with w>v then
  - v forwards w to clockwise neighbor
  - v *decides* not to be the leader
- else if v receives its own ID v then
  - v *decides* to be the leader

Evaluation

Time Complexity

Number of rounds (for asynchronous, assume max delay of one unit).

Message Complexity

Number of messages sent.

„Local Complexity“

Local computations...

For our algorithm?!

Clockwise Algorithm

Theorem

Algo is correct, time complexity O(n), message complexity O(n^2).

Proof idea?

Correctness: Let z be max ID. No other node can swallow z's ID, so z will get the message back. So z becomes leader. Every other node declares non-leader when forwarding z (the latest!).

Message complexity: Each node forwards at most n messages (n IDs in total).

Time complexity: Message circles around cycle (depending on model, at most twice: once to wake up z, and then until z becomes leader).

QED

Can we do better?!
Time? Messages? ...

Radius Growth

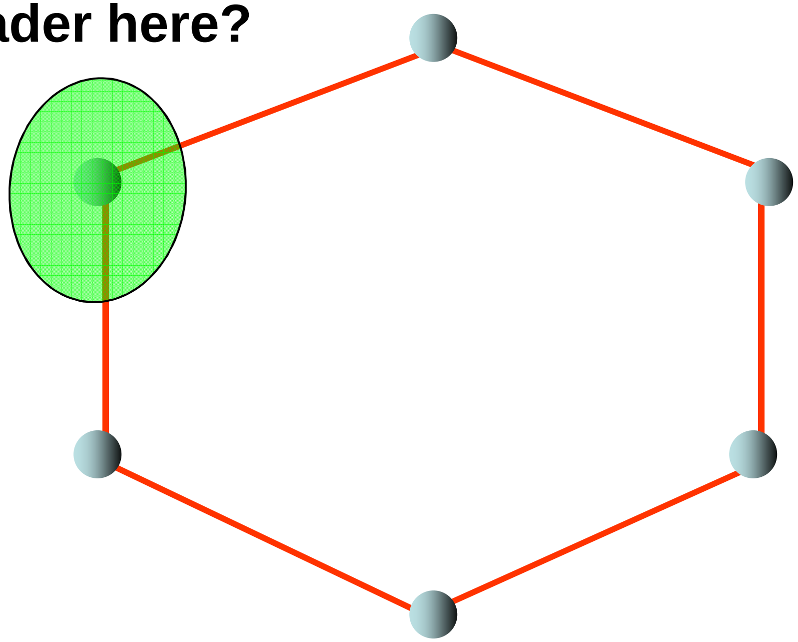
Algorithm Radius Growth

each node v does the following:

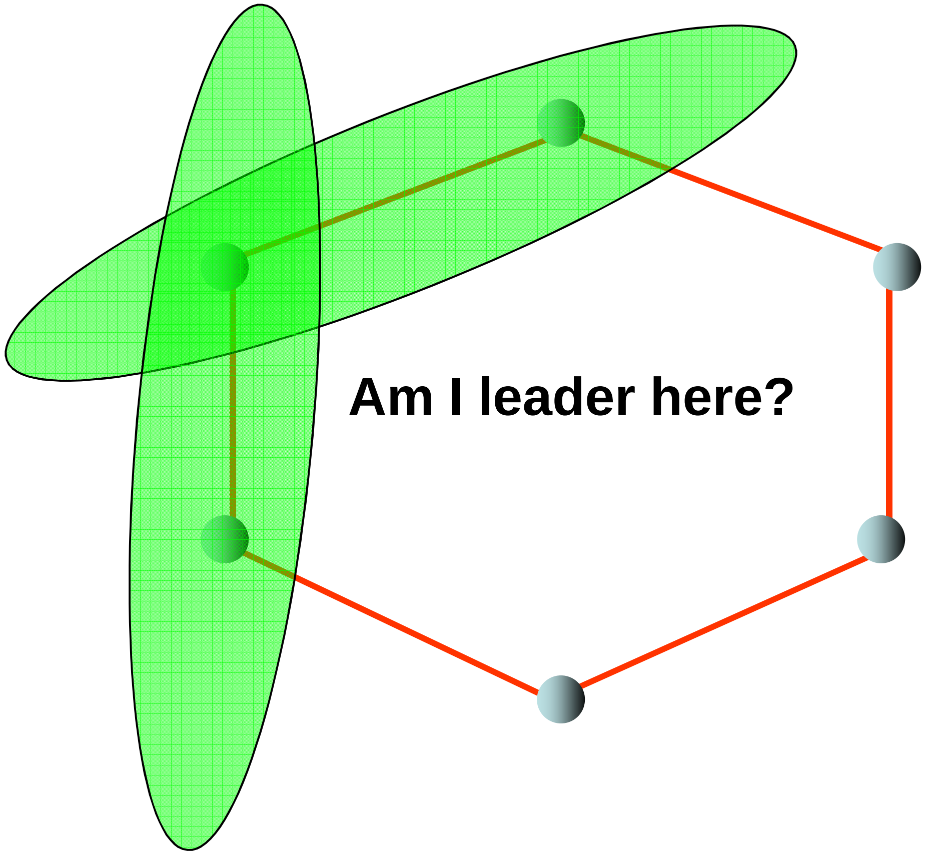
- Initially, all nodes are active (can still become leader)
- Whenever a node v sees a message with w>v, it *decides* not to be a leader and becomes passive
- Active nodes search in an exponentially growing neighborhood (clockwise and counterclockwise) for nodes with higher IDs by sending out *probe messages*: a probe includes sender‘s ID, a leader bit saying whether original sender can still become a leader, and TTL (initially =1).
- All nodes w receiving a probe decrement TTL and foward to next neighbor; if w‘s ID is larger than original sender‘s ID, the leader bit is set to zero. If TTL=0, return message to sender (reply msg) including leader bit.
- If leader bit is still 1, double the TTL, and two new probes are sent (for both neighbors); otherwise node becomes passive.
- If v receives its own probe message (not the reply): it becomes leader.

Radius Growth

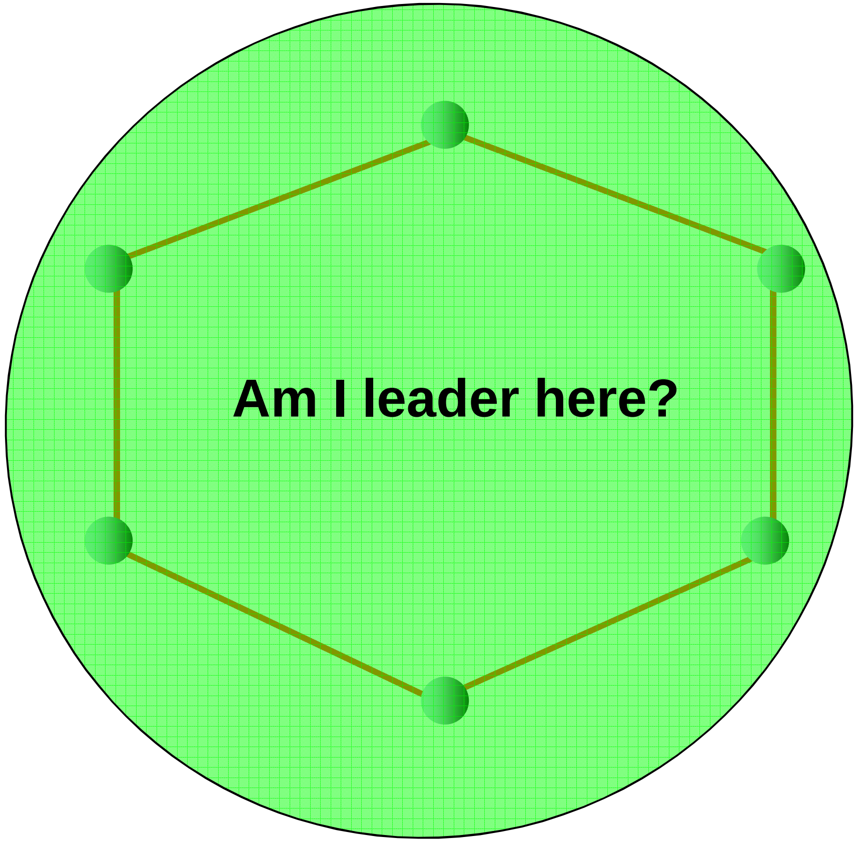
Am I leader here?



Radius Growth



Radius Growth



How to analyze?
Complexities?

Radius Growth

Theorem

Algo is correct, time complexity O(n), message complexity O(n log n).

Proof idea?

Correctness: Like clockwise algo.

Time complexity: O(n) since node with max identifier sends messages with round trip times 2, 4, 8, ..., 2^k with $k \in O(\log n)$. The sum constitutes a geometric series and is hence linear in n.

Message complexity: Only one node can survive phase p that covers a distance of 2^p. So less than $n/2^p$ nodes are active in round p+1. Being active in round p costs roughly 2^p messages, so it's around O(n) per round over all active nodes. As we have a logarithmic number of phases, the claim follows.

QED

Can we do better?!

Or how can we prove that we cannot?

Lower bounds!

Lower Bound (1)

Take-Away

In message passing systems, lower bounds can often be proved by arguing about messages that need to be exchanged!

Concepts:

1. Generally, we need some definitions to characterize the class of algorithms for which the lower bound holds.
2. Moreover, in distributed systems, a (hypothetical) scheduler determines sequence of events...

Execution

An execution of a distributed algorithm is a list of events, sorted by time. An event is a record

(time, node, type, message)

where type is „send“ or „receive“.

Lower Bound (2)

Assumptions:

- Asynchronous ring: nodes wake up at arbitrary times but always when receiving a packet
- nodes have IDs, and node with max ID should become leader
- every node must know ID of leader
- uniform algorithm: n is not known
- arbitrary scheduler but links are FIFO

For our lower bound proof, we define the concept of *open schedules*:

Open Schedule

Schedule chosen by scheduler. Open if there is an open edge in the ring. Edge is *open* if no message traversing edge has been received so far.

Some Intuition...

Open Schedule

Schedule chosen by scheduler. Open if there is an open edge in the ring. Edge is *open* if no message traversing edge has been received so far.

Intuitively: Open schedule = endpoints have not heard anything from nodes on this edge, protocol *cannot stop* yet as it may hide critical infos on the leader!

We want to show that there exists a *bad schedule* which requires lots of messages until a leader is elected. To achieve this, we compute an open schedule inductively.

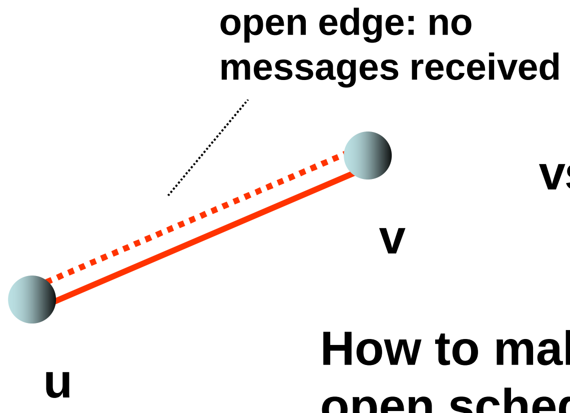
Lower Bound by Induction

Proof by induction:

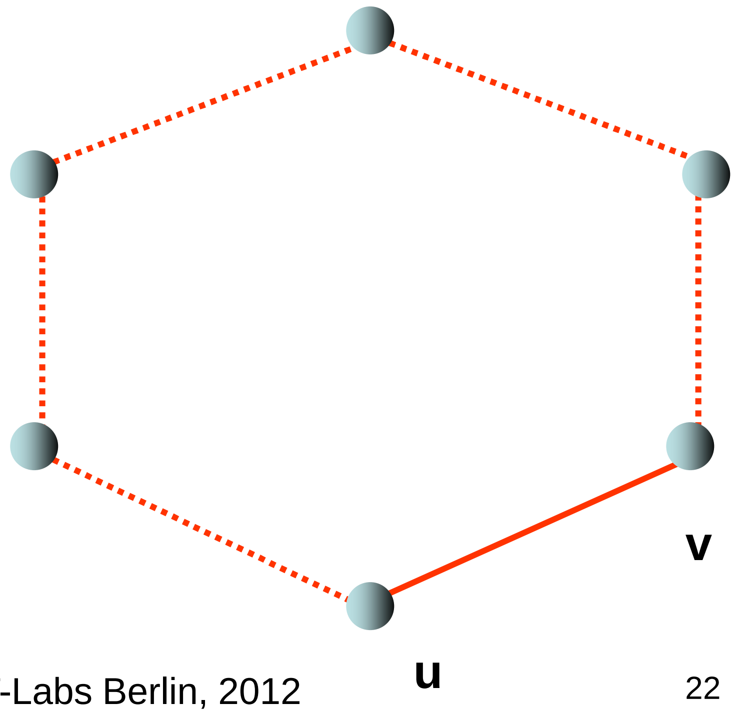
Lemma: 2-node Ring

Given a ring R with two nodes, we can construct an open schedule in which at least one message is received. The nodes cannot distinguish this schedule from one on a larger ring with all other nodes being located where the open edge is.

Proof of Lemma: u and v cannot distinguish between the two scenarios!



vs



How to make an open schedule?

Proof of Lemma: Open Schedule

Lemma: 2-node Ring

Given a ring R with two nodes, we can construct an open schedule in which at least one message is received. The nodes cannot distinguish this schedule from one on a larger ring with all other nodes being where the open edge is.

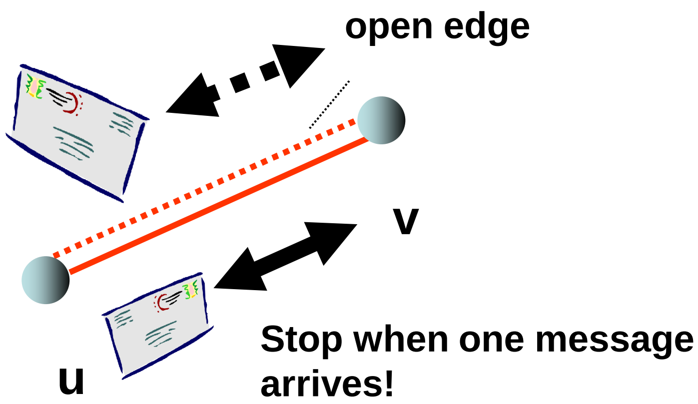
Open schedule for 2-node ring?

In any leader election algorithm, the two nodes must learn about each other! We stop execution when first message is received (on whatever link).

We can do this because it‘s an asynchronous world (no simultaneous arrivals)...

So other edge is open:

Nodes don‘t know, is it an edge, or is it more?



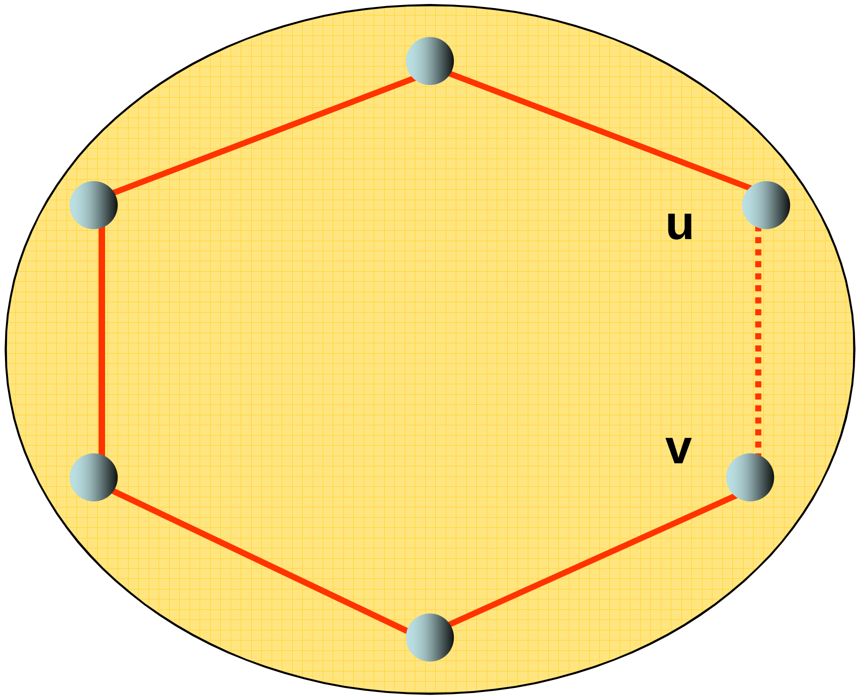
QED

Open Schedules for Larger Rings?

n-node Ring

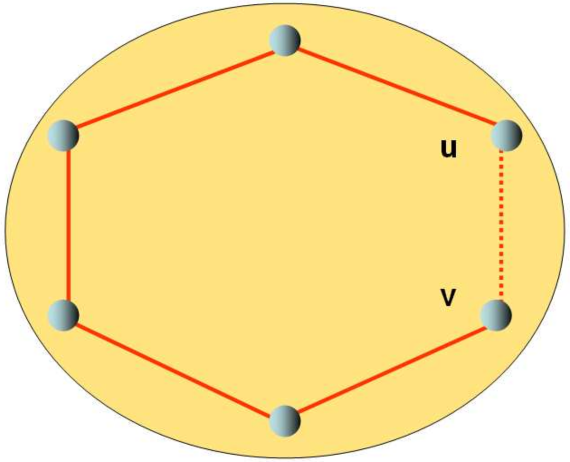
By gluing together two rings of size n/2 for which we have open schedules, an open schedule can be constructed on a ring of size n. Let M(n/2) denote the number of messages used in each of these schedules by some algorithm ALG. Then, in the entire ring 2M(n/2)+n/4 messages have to be exchanged to solve leader election.

Proof? Open schedule?



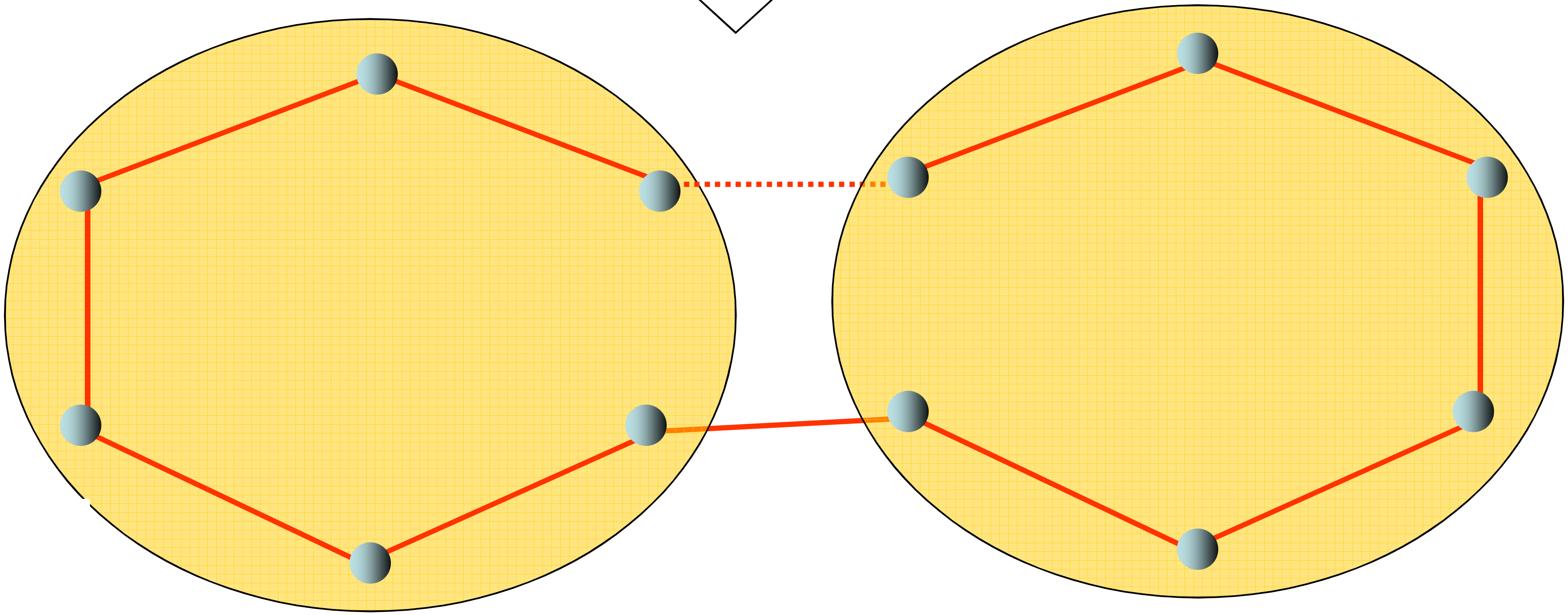
Assume ALG needs M(n/2) messages here...

... how many for the whole ring?



Idea: take two times smaller ring and „close“ one edge...

Open schedule for larger ring?

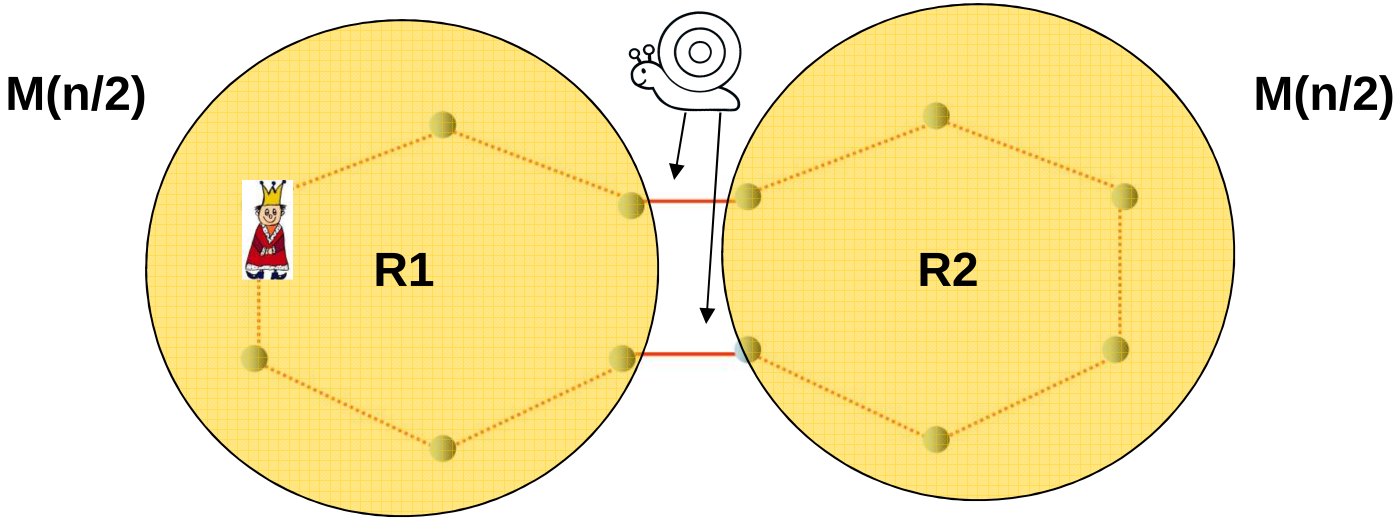


Proof of Lemma: By Induction

- Consider the ring of size n and divide it in two „subrings“ R1 and R2. As long as no message comes from outside, nodes cannot distinguish these two rings from two rings of size n/2. (Just delay messages accordingly: all other messages of algorithm are sent.)

- So nodes exchange 2*M(n/2) messages (induction hypothesis) in the subrings before learning anything about the other subring. Wlog assume R1 has max ID. So each node in R2 must learn that ID, which requires at least n/2 message receptions.

- So there must be an edge connecting the two rings that „produces“ (= triggers, but not necessarily transmits!) at least n/4 messages.
  Schedule/close this edge and leave other open... => open schedule for larger ring! And enough messages! ☺



Open Schedules for Larger Rings?

Theorem

Any algo needs at least Ω(n log n) messages.

Proof by induction: Claim follows from maths...

$$\begin{aligned} M(n) &= 2 \cdot M\left(\frac{n}{2}\right) + \frac{n}{4} \\ &\geq 2 \cdot \left(\frac{n}{8}\left(\log\frac{n}{2} + 1\right)\right) + \frac{n}{4} \\ &= \frac{n}{4}\log n + \frac{n}{4} = \frac{n}{4}(\log n + 1) \end{aligned}$$

So we are optimal.

Can we do better? ☺

Breaking the Lower Bound ☺

Take-Away

In synchronous systems, not receiving a message is also information!

Idea for message complexity n? E.g., find minimum ID in environment where nodes have unique but arbitrary integer IDs (but n known)...

Sync Leader Election

each node v does the following:

- Divide time into phases of n steps (leaves time for lower-ID nodes to broadcast...)
- If phase = v and did not get a message:
  - v becomes leader
  - v sends „I am leader!“ to everybody!

Breaks message lower bound but we may wait long!
Runtime O(n*minID)? What is the time – message tradeoff?

Literature for further reading:

- Attiya/Welch (Alg. 3.1 for example)
- Peleg‘s book (as always ☺)

End of lecture