Network Optimization by Randomization: Exercise Distributed Algorithms and Social Networks

1 Vertex Coloring

In the lecture, a simple distributed algorithm ("Reduce") which colors an arbitrary graph with $\Delta + 1$ colors in n synchronous rounds was presented (Δ denotes the largest degree, n the number of nodes of the graph).

a) What is the message complexity, i.e., the total number of messages the algorithm sends in the worst case?

b) Does the algorithm also work in an asynchronous environment? If yes, formulate the asynchronous equivalent to the algorithm, if no, describe why.

2 License to Match

In preparation of a highly dangerous mission, the participating agents of the gargantuan Charlottenburger secret service (CSS) need to work in pairs of two for safety reasons. All members in the CSS are organized in a tree hierarchy. Communication is only possible via the official channel: an agent has a secure phone line to his direct superior and a secure phone line to each of his direct subordinates.

Assume that at some time t_0, a subset of agents (from the whole tree hierarchy) learn that they need to take part in a mission. The goal for each "activated" agent is then to find a partner (chosen arbitrarily from the CSS tree), by using the official communication links mentioned above.

a) Devise an algorithm that will match up a participating agent with another participating agent given the constrained communication scenario. A "match" consists of an agent knowing the identity of his partner and the path in the hierarchy connecting them. Assume that there is an even number of participating agents so that each one is guaranteed a partner. Furthermore, observe that—in the case of an emergency where they lose contact—the phone links connecting two paired-up agents need to remain open at all times. Therefore, you cannot use the same link (i.e., an edge) twice when connecting agents with their partners.

b) What are the time and message (i.e., "phone call") complexities of your algorithm?

3 Diameter of the Augmented Grid

Recall the network from the lecture where nodes were arranged in a grid and each node had an additional directed link to an uniformly and independently at random drawn node in the network (i.e., $\alpha = 0$). In the lecture, a proof of the fact that such a network has diameter $O(\log n)$ w.h.p. was sketched. We will now fill in details.

a) Show that $O(cn/\log n)$ many nodes are enough to guarantee with high probability that at least one of their random links connects to a given set of $\Omega(\log^2 n)$ nodes. Prove this (i) by direct calculation and (ii) using Chernoff's bound.

Hint: Use that $1 - p \leq e^{-p}$ for any p.

b) Suppose for some node set S we have that $|S| \in \Omega((\log n)^2) \cap o(n)$ and denote by H the set of nodes hit by their random links. Prove that H and together with its grid neighbors contains w.h.p. $(5 - o(1))|S|$ nodes!

Hint: Observe that *independently* of all previous random choices, each new link has at least a certain probability p of connecting to a node whose complete neighborhood has not been reached yet. Then use Chernoff's bound on the sum of $|S|$ many variables.

c) Infer from **b)** that starting from $\Omega(\log^2 n)$ nodes, with each hop the number of reached nodes w.h.p. more than doubles, as long as we have still $O(n/\log n)$ nodes (regardless of the constants in the O-notation).

Hint: Play with the constant c in the definition of w.h.p. and use the union bound.

d) Conclude that the diameter of the network is w.h.p. in $O(\log n)$.