Approximate Duality of Multicommodity Multiroute Cuts and Flows: Multilevel Ball Growing

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Berlin, July 20, 2011

Multicommodity Multiroute Cut

Input

- graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}_+$
- k commodities specified by pairs $(s_1, t_1), \ldots, (s_k, t_k) \subseteq V^2$
- integer parameter h

Output

- a subset of edges $F \subseteq E$ such that
  no pair (s_i, t_i) is connected by h edge-disjoint paths in $(V, E \setminus F)$

Objective

- find a cut of minimum size, that is,
  minimize $\sum_{e \in F} c(e)$ subject to *"F is an h-route cut"*

Linear Programming Formulation

Notation

- $\mathcal{Q}_i$ - the set of all tuples of h edge-disjoint paths between s_i and t_i
- $\mathcal{Q} = \bigcup_{i=1}^{k} \mathcal{Q}_i$

Minimum Cut LP(1)

$$\min \sum_{e \in E} c(e) \cdot x(e)$$

$$\sum_{e \in q} x(e) \geq 1 \quad \forall q \in \mathcal{Q}$$

$$x(e) \in \{0, 1\} \quad \forall e \in E$$

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Maximum Flow

$$\max \sum_{q \in \mathcal{Q}} f(q)$$

$$\sum_{q \in \mathcal{Q} : e \in q} f(q) \leq h \cdot c(e) \quad \forall e \in E$$

$$f(q) \geq 0 \quad \forall q \in \mathcal{Q}$$

Observation

- The two LP's are dual.

$\Rightarrow$ By LP-duality, an upper bound on the integrality gap of LP(1) implies an approximate duality of the MM-flows and MM-cuts.

Better Linear Programming Formulation

- $\mathcal{P}_i$ - the set of all paths between s_i and t_i

Minimum Cut LP(2)

$$\min \sum_{e \in E} c(e)x(e)$$

$$\begin{aligned} \sum_{e \in p} (x(e) + x_i(e)) &\geq 1 && \forall i \in [k], p \in \mathcal{P}_i \\ \sum_{e \in E} x_i(e) &\leq h - 1 && \forall i \in [k] \\ x(e), x_i(e) &\in \{0, 1\} && \forall e \in E, \forall i \in [k] \end{aligned}$$

Comparison with LP(1)

for $h = 1$, LP(1)=LP(2)

\+ stronger

\+ induces easier structure (simple paths vs. tuples of edge-disjoint paths)

\- no clear relation to the maximum multiroute flow

New Results

Theorem

- *The integrality gap of the LP(2) is* $O(2^{2h} \log^3 k)$.
- *The LP(2) is stronger than the LP(1) by a factor of h at most.*

Corollary

- *Approximate Duality of multicommodity multiroute cuts and flows.*

Previous Results

Duality of Flows and Cuts (Ford and Fulkerson, 1956)

$$\text{max-flow} = \text{min-cut}$$

Approximate Duality for Multicommodity Flows (FOCS'88, STOC'93)

$$\text{max-flow} \leq \text{min-cut} \leq O(\log k)\text{max-flow}$$

Approximate Duality for Multiroute Flows

- Single source, uniform capacities: factor $2h$ (SODA 2008)
- General setting
  - $h = 2$: factor $O(\log^2 n \log k)$ (Chekuri and Khanna, ICALP 2008)
  - $h = 3$: factor $O(\log^3 k)$ (K. and Scheideler, STACS 2011)
- Related results (Barman and Chawla, SODA 2010)
  - integrality gap of LP(2) is $O(\log^2 k)$
  - in case of single commodity - the gap is 2

High Level Description of Rounding Algorithms

Algorithm for Single Source Cut

Input: graph G, fractional solution $x, x_1, \ldots, x_k \in \mathcal{R}^{|E|}$ of the LP(2)
$F := \emptyset$
while exists h-connected pair (s, t_i) **do**
  with the help of x and x_i, find a cheap cut F_i for (s, t_i) in G
  $F := F \cup F_i$
  update G (e.g., $G := G \setminus F_i$)
return F

This presentation

- How to obtain from a fractional solution of the LP(2) the cheap cuts $F_1, \ldots, F_k$.

$\Rightarrow$ Upper bound on the integrality gap of the LP(2).

$\Rightarrow$ Approximate duality.

Finite Metric Spaces and the LPs

Shortest path metric $h = 1$

$d(u, v) =$ length of the shortest path between u and v in G, with respect to x

Universal metric for all commodities - used by Garg et al.

Potential metric $h \geq 2$

Personal metric for each commodity:

$$d_i(u, v) = |y_i(u) - y_i(v)|$$

where $y_i(w) =$ length of the shortest path between t_i and w, with respect to $x + x_i$

Basic properties

- $d_i(s, t_i) \geq 1$
- $\forall uv \in E : \; d_i(u, v) \leq x(u, v) + x_i(u, v)$

Balls, Cuts and Volume for $h = 1$

Focus on iteration i

Key notions

For $r \in [0, 1/2]$ define:

$$\begin{aligned} \text{ball} \quad B(r) &= \{u \in V \mid d(t_i, u) \le r\} \\ \text{cut} \quad \delta(r) &= \{uv \in E \mid u \in B(r), v \notin B(r)\} \\ \text{cut size} \quad C(r) &= \sum_{e \in \delta(r)} c(e) \\ \text{volume} \quad V(r) &= \frac{\phi}{k} + \int_0^r C(\rho)\, \mathrm{d}\rho \quad \text{where } \phi = \mathrm{OPT}(LP(2)) \end{aligned}$$

Observation

$$V'(r) = C(r)$$

Ball Growing Technique for $h = 1$

Key ideas

- Allow only cuts around balls - $\delta(r)$.
- Charge the cost $C(r)$ of the cut $\delta(r)$ to the volume $V(r)$ of the ball $B(r)$.
- The quantity $\phi = \sum_{e \in E} x(e)c(e)$ - volume of the entire graph - is a lower bound on the optimal cut.
- The cut $\delta(r)$ separates the graph into two independent parts.
- Always remove the ball $B(r)$ from the graph.

Lemma (Garg et al., 1993)

There always exists a radius $r < 1/2$ such that

$$C(r) \leq \log k \cdot V(r)$$

Note: No commodity fits in a ball with radius $r < 1/2$.

Ball Growing and the Calculus Behind

Lemma (Ball Growing)

If two functions f and g defined on $\mathcal{R}$ satisfy

- f *is a nondecreasing function on* $\mathcal{R}$ *and* $f > 0$,
- f *is differentiable on* $\mathcal{R}$,
- $\forall r \in \mathcal{R},\ g(r) \leq f'(r)$,

then there exists $r \in \mathcal{R}$ *such that*

$$g(r) \leq \frac{1}{|\mathcal{R}|} \log \frac{f(r_{\max})}{f(r_{\min})} \cdot f(r)\ .$$

Lemma (Garg et al., 1993)

The functions volume $f(r) = V(r)$ *and cost* $g(r) = C(r)$ *satisfy for* $\mathcal{R} = [0, 1/2)$ *the assumptions of the lemma and* $f(r_{\max})/f(r_{\min}) \leq k$.

$\Rightarrow$ the radius r from the lemma provides a cheap cut for $h = 1$

Ball Growing for Multiroute Cuts

Difficulty for $h \geq 2$

- Cuts must be redefined - no approximation algorithm with finite approximation ratio can afford to perform $\delta(r)$ cuts.

How to fix it?

- Think about edge e as about a segment consisting of two parts: x_i-part of length $x_i(e)$ and x-part of length $x(e)$.
- Only edges crossing the border of $B(r)$ in x-part belong to cut.
- Only x-parts of edges contribute to the volume.
- Both parts of every edge - $x(e) + x_i(e)$ - contribute to distances.

Cuts and Volume Revisited

Assumption

- The x_i-part precedes (on the way from t_i) the x-part.

Key notions

$$\begin{array}{rrcl}
\text{cut} & \delta_x(r) & = & \{uv \in E \mid y_i(u) + x_i(u,v) \leq r < y_i(v)\} \\
 & \delta_{x_i}(r) & = & \{uv \in E \mid y_i(u) \leq r < y_i(u) + x(u,v)\} \\
\text{cut size} & C(r) & = & c(\delta_x(r)) \\
\text{volume} & V(r) & = & \phi/k + \int_{\rho \in [0,r]} C(\rho) \mathrm{d}\rho
\end{array}$$

Observation

- $V'(r) = C(r)$
- $\delta_x(r) \cup \delta_{x_i}(r) = \delta(r)$
- *If $|\delta_{x_i}(r)| \leq h - 1$, then $\delta_x(r)$ is an h-route cut.*

Ball Growing Technique for $h = 2$

Basic Notation

- A radius $r \in [0, 1]$ is forbidden if $|\delta_{x_i}(r)| > h - 1$.
- $\mathcal{R} = \{r \in [0, 1] \mid \delta_{x_i} \leq h - 1\}$

Lemma (Forbidden Radii)

The size of the set $\mathcal{R}$ of good radii is at least $1/h$.

Proof:

- $\mu = |\{r \in [0, 1] \mid |\delta_{x_i}(r)| \geq h\}|$
- Recall that $\sum_{e \in E} x_i(e) \leq h - 1$.
- Therefore, $h \cdot \mu \leq \sum_{e \in E} x_i(e) \leq h - 1$ and hence, $\mu \leq 1 - 1/h$.

Consequence

Ball Growing Lemma: good radius r with $c(\delta_x(r)) \leq h \log k \cdot V(r)$.

Ball Growing Technique for $h = 2$, cont'd

Single Source Case

- For each t_i, find ball $B(r)$ with h-route cut $c(\delta_x(r)) \leq h \log k \cdot V(r)$.
- All t_j's with $t_j \in B(r)$ are done as well.
- For all t_j's with $t_j \notin B(r)$: no path from t_j to s through $B(r)$ as at most one edge is leaving $B(r)$, so $B(r)$ can be removed.

Multiple Source Case

Problem: Many source-destination pairs in ball $B(r)$ of t_i.

Solution:

- Recall that $\mathcal{R} = \{r \in [0,1] \mid |\delta_{x_i}(r)| \leq h - 1\}$ and that $|\mathcal{R}| \geq 1/h$.
- Cut $\mathcal{R}$ into $\mathcal{R}_1$ and $\mathcal{R}_2$ of size $1/(2h)$.
- Grow balls from t_i and s_i with radii r_1 and r_2.
- Charge volume of that ball with the smaller # (s_j, t_j)-pairs.
- Continue recursively in ball and outside part of G.

Ball Growing Technique for $h = 3$

Another Difficulty for $h = 3$

Even if we know how to find a cheap 3-route cut, the 3-route cut does not split the 3-route cut problem into independent problems.

Reason: Some t_j can have a path to s_j leading through the ball of (s_i, t_i).

How to fix it? Multilevel ball growing

- Allow edges to be charged several times, yet
- Ensure that no edge is charged too often. How???
- Maintain a counter level $\ell(e)$ for every edge - how many times e was charged for some cut.
- When growing a new ball, avoid edges with high level.

Multilevel Ball Growing

Pessimistic scenario

- Construct a ball for commodity 1 and perform corresponding cut.
- Construct a new ball for commodity 2, perform corresponding cut; accidently, the new ball contains the previous ball inside.
- Construct a new ball for commodity 3, perform corresponding cut; accidently, the new ball contains the previous ball inside again.
- etc.

$\Rightarrow$ The level of some edges is k at the end - too much.

Idea

- Whenever a ball of level ℓ is included in a new ball, make sure there are at least two balls of level ℓ that are included.

$\Rightarrow$ The maximum level of an edge is $\log k$.

Multilevel Ball Growing, cont'd

D - set of balls from previous iterations

Partitioning of D in iteration i

- D_1 - for every level ℓ, the ball of level ℓ that is the closest to t_i
- $D_2 = D \setminus D_1$

Good radii - first attempt

$$\mathcal{R} = \{r \in [0,1] \mid |\delta_{x_i}(r)| \leq h - 1 \textit{ and } \delta_x(r) \cap E(D_1) = \emptyset\}$$

Note that $V'(r) = C(r)$, for $V(r) = \phi/k + \int_{\rho \in \mathcal{R} \cap [0,r]} C(\rho) \mathrm{d}\rho$
$\Rightarrow$ Ball growing lemma applies:

$$\exists r \in \mathcal{R} \text{ s.t. } C(r) \leq \frac{1}{|\mathcal{R}|} \log k \cdot V(r)$$

However: how large is $\mathcal{R}$?? Might be too small $\Rightarrow$ bound too weak.

How to make the set of good radii large?

Restricted Diameter Lemma

- Entry edges $in_E(H)$ of ball B: edges that connect B to rest of G.
- Entry nodes $in_V(H)$ of ball B: outside nodes of entry edges.

Lemma: If x-diameter of entry nodes of ball B is at least $1/(2h \log k)$ then the cost of the mincut of the entry nodes in H is at most $2h \log k \cdot Vol(B)$.

- If lemma applies to ball B, B is cut into two halves with at most one entry edge, so these halves can be removed from G.
- So we are only left with balls B with small diameter.
- Thus, diameters of all balls in D_1 sum up to at most $1/(2h)$.
- This implies that size of good radii shrinks from at least $1/h$ to at least $1/(2h)$.

Still, general multi-source 3-route cut problem very tricky...

How to make the set of good radii large? Case $h \geq 3$.

Definition: The radius r is blocked by H if $\delta_x(r) \cap E(H) \neq \emptyset$.

Lemma

For every $H \in D_1$ there exists a subset $F_H \subseteq E(H)$ such that the measure of the set of radii blocked by $H \setminus F_H$ is small and

$$\sum_{e \in F_H} c(e) \leq O(2^{2h} \log k) \sum_{e \in E(H)} x(e) c(e) \ .$$

- For $h = 3$ - F_H is a simple cut separating the two entry points of the ball H; by doing this, H can be removed from D and G.
- For $h > 3$ - substantially more complicated: we have to increase the level for each $e \in E(H)$ to reflect that H is charged for the removal of F_H.

$\Rightarrow$ New problem: how to keep the levels under control? Credit ...

Restricted Structures

Basic definitions

- A restricted structure H is an induced subgraph of G.
- Entry edges $in_E(H)$ of H - edges that connect H to the rest of G.
- The level of a restricted structure H is

$$\ell(H) = \max\{\ell(e) \mid e \in E(H) \setminus \bigcup_{H' \in D(H)} E(H')\} . \tag{1}$$

  where $D(H) = \{H' \in D \mid H' \subsetneq H\}$.
- The credit of a restricted structure H is

$$\phi(H) = 2^{2|in_E(H)|+\ell(H)} . \tag{2}$$

- For each level ℓ, put to D_1 the minimum number of restricted structures of level ℓ whose credit sums up to at least $2h^2 2^{2h+\ell}$, starting with the closest to t_i.

Restricted Structures, cont'd.

Invariant

At the end of iteration i the structures in D form a laminar family such that for all $H \in D$

$$|in_E(H)| \leq h - 1 \;, \tag{3}$$

$$\sum_{H \in D} \phi(H) \leq i \cdot 2^{2h} \;. \tag{4}$$

Lemma

If the Invariant holds, then

- *at the end of iteration i, $\ell(e) \leq 2h + \log i$ for every edge e,*
- *for each level ℓ, $\sum_{H \in D_1 : \ell(H) = \ell} |in_E(H)| \leq h^2 2^{2h-1}$.*

Many Good Radii - Crucial Lemma

Lemma

For every $H \in D_1$ there is a subset $F_H \subseteq E(H)$ and a subset $I_H \subseteq R_H$ so that for $H' = H \setminus F_H$,

$$\int_{r \in R_H \setminus I_H} |\mathrm{mincut}_{H'}(r)| dr \leq \sum_{e \in E(H)} x_i(e) \,, \tag{5}$$

$$\sum_{e \in F_H} c(e) = O(h^6 2^{2h}) \sum_{e \in E(H)} x(e) c(e) \,, \tag{6}$$

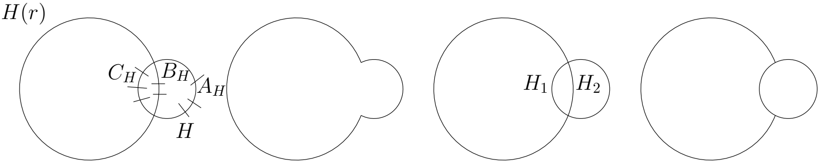
$$|I_H| \leq 1/(h^3 2^{2h} \log k) \,. \tag{7}$$

Other Difficulties with the Balls

Laminar sets

For previous lemma, the set D has to satisfy the laminar property: for any two $H_1, H_2 \in D$: $H_1 \subset H_2$, or $H_2 \subset H_1$, or $H_1 \cap H_2 = \emptyset$.

To maintain the laminar property, a special care must be given to old balls intersecting with the border of the new ball:



$|A_H| \leq |B_H|$

$|B_H| < |A_H|$ and $|B_H| < |C_H|$

$|C_H| \leq |B_H| < |A_H|$

Conclusion

Disclaimer

- Plenty of details skipped ...

Open problems

- Improve the upper bound on the factor in the approximate duality.
- Improve the lower bound on the factor in the approximate duality.
- Find a better approximation algorithm for the multicommodity multiroute cut problem.
- Apply the multilevel ball growing technique to other problems ...