Comparing alternatives

Prof Anja Feldmann
based on slides by David J. Lilja
Comparing alternatives

- ANOVA
  - Analysis of Variance
- Partitions total variation in a set of measurements into
  - Variation due to real differences in alternatives
  - Variation due to errors
Comparing two alternatives

1. *Before-and-after*
   Did a change to the system have a statistically significant impact on performance?

2. *Non-corresponding measurements*
   Is there a statistically significant difference between two different systems?
Before-and-after comparison

- Assumptions
  - Before-and-after measurements are not independent
  - Variances in two sets of measurements may not be equal
  → Measurements are related

- Use *mean of differences*
Before-and-after comparison

<table>
<thead>
<tr>
<th>Measurement (i)</th>
<th>Before ($b_i$)</th>
<th>After ($a_i$)</th>
<th>Difference ($d_i = b_i - a_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>85</td>
<td>86</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>83</td>
<td>88</td>
<td>-5</td>
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<tr>
<td>3</td>
<td>94</td>
<td>90</td>
<td>4</td>
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<td>4</td>
<td>90</td>
<td>95</td>
<td>-5</td>
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<td>5</td>
<td>88</td>
<td>91</td>
<td>-3</td>
</tr>
<tr>
<td>6</td>
<td>87</td>
<td>83</td>
<td>4</td>
</tr>
</tbody>
</table>
Before-and-after comparison

Mean of differences $\bar{d} = -1$
Standard deviation $s_d = 4.15$

- From mean of differences, appears that change reduced performance.
- However, standard deviation is large.
95% Confidence interval for mean of differences

- $c_{1,2} = [-5.36, 3.36]$
- Interval includes 0

→ With 95% confidence, there is no statistically significant difference between the two systems.
Noncorresponding measurements

- No direct correspondence between pairs of measurements
- *Unpaired* observations
- $n_1$ measurements of system 1
- $n_2$ measurements of system 2
Confidence interval for difference of means

1. Compute means
2. Compute difference of means
3. Compute standard deviation of difference of means
4. Find confidence interval for this difference
5. No statistically significant difference between systems if interval includes 0
OS example

- Initial operating system
  - \( n1 = 1,300,203 \) interrupts (3.5 hours)
  - \( m1 = 142,892 \) interrupts occurred in OS code
  - \( p1 = 0.1099 \), or 11\% of time executing in OS

- Upgrade OS
  - \( n2 = 999,382 \)
  - \( m2 = 84,876 \)
  - \( p2 = 0.0849 \), or 8.5\% of time executing in OS

- Statistically significant improvement?
OS example (2.)

- $p = p_1 - p_2 = 0.0250$
- $s_p = 0.0003911$
- 90% confidence interval
  - $(0.0242, 0.0257)$
- Statistically significant difference?
Important points

- Use confidence intervals to determine if there are statistically significant differences
  - Before-and-after comparisons
    - Find interval for mean of differences
  - Noncorresponding measurements
    - Find interval for difference of means

- If interval includes zero
  → No statistically significant difference
Comparing > two alternatives

- Naïve approach
  - Compare confidence intervals
One-factor Analysis of Variance (ANOVA)

- Very general technique
  - Look at total variation in a set of measurements
  - Divide into meaningful components
- Also called
  - One-way classification
  - One-factor experimental design
One-factor Analysis of Variance (ANOVA)

- Separates total variation observed in a set of measurements into:
  1. Variation within one system
    - Due to random measurement errors
  2. Variation between systems
    - Due to real differences + random error

- Is variation(2) statistically > variation(1)?
ANOVA

- Make \( n \) measurements of \( k \) alternatives
- \( y_{ij} = \text{ith measurement on jth alternative} \)
- Assumes errors are:
  - Independent
  - Gaussian (normal)
Measurements for all alternatives

<table>
<thead>
<tr>
<th>Measurements</th>
<th>Alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$y_{11}$</td>
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<td>Col mean</td>
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<tr>
<td>Effect</td>
<td>$\alpha_1$</td>
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</table>
Column means:
Average performance of one alternative

<table>
<thead>
<tr>
<th>Measurements</th>
<th>1</th>
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Error:
Deviation from column mean

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<tr>
<th>Measurements</th>
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Overall mean:
Average performance of all alternatives

<table>
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<tr>
<th>Measure</th>
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</table>
Effect:
Deviation from overall mean

<table>
<thead>
<tr>
<th>Measurem ents</th>
<th>1</th>
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<td>Effect</td>
<td>$\alpha_1$</td>
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<td>$\alpha_j$</td>
<td>...</td>
<td>$\alpha_k$</td>
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</tbody>
</table>
Effects and errors

- **Effect** is distance from overall mean
  - Horizontally across alternatives
- **Error** is distance from column mean
  - Vertically within one alternative
  - Error across alternatives, too
- Individual measurements are then:

\[ y_{ij} = \bar{y}_{..} + \alpha_j + e_{ij} \]
Sum of squares of differences

- \( SST \) = differences between each measurement and overall mean
- \( SSA \) = variation due to effects of alternatives
- \( SSE \) = variation due to errors in measurements

\[
SST = SSA + SSE
\]
Sum of squares of differences

\[ SSA = n \sum_{j=1}^{k} \left( \bar{y}_j - \bar{y}_{..} \right)^2 \]

\[ SSE = \sum_{j=1}^{k} \sum_{i=1}^{n} \left( y_{ij} - \bar{y}_j \right)^2 \]

\[ SST = \sum_{j=1}^{k} \sum_{i=1}^{n} \left( y_{ij} - \bar{y}_{..} \right)^2 \]
ANOVA – Fundamental idea

Separates variation in measured values into:

1. Variation due to effects of alternatives
   - SSA – variation across columns
2. Variation due to errors
   - SSE – variation within a single column

If differences among alternatives are due to real differences,
   - SSA should be statistically > SSE
Comparing SSE and SSA

- Simple approach
  - \( SSA / SST \) = fraction of total variation explained by differences among alternatives
  - \( SSE / SST \) = fraction of total variation due to experimental error
- But is it statistically significant?
Comparing variances

- Use F-test (statistics) to compare ratio of variances
- If $F_{\text{computed}} > F_{\text{table}}$
  → We have $(1 - \alpha) \times 100\%$ confidence that variation due to actual differences in alternatives, SSA, is statistically greater than variation due to errors, SSE.
## ANOVA Example

<table>
<thead>
<tr>
<th>Measurements</th>
<th>Alternatives</th>
<th>Overall mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0.0972</td>
<td>0.1382</td>
</tr>
<tr>
<td>2</td>
<td>0.0971</td>
<td>0.1432</td>
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<tr>
<td>3</td>
<td>0.0969</td>
<td>0.1382</td>
</tr>
<tr>
<td>4</td>
<td>0.1954</td>
<td>0.1730</td>
</tr>
<tr>
<td>5</td>
<td>0.0974</td>
<td>0.1383</td>
</tr>
<tr>
<td><strong>Column mean</strong></td>
<td>0.1168</td>
<td>0.1462</td>
</tr>
<tr>
<td><strong>Effects</strong></td>
<td><strong>-0.1735</strong></td>
<td><strong>-0.1441</strong></td>
</tr>
</tbody>
</table>

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Conclusions from example

- SSA/SST = 0.7585/0.8270 = 0.917
  → 91.7% of total variation in measurements is due to differences among alternatives
- SSE/SST = 0.0685/0.8270 = 0.083
  → 8.3% of total variation in measurements is due to noise in measurements
- Computed $F$ statistic > tabulated $F$ statistic
  → 95% confidence that differences among alternatives are statistically significant.
Contrasts

- ANOVA tells us that there is a statistically significant difference among alternatives
- But it does not tell us where difference is
- Use method of contrasts to compare subsets of alternatives
  - A vs B
  - \{A, B\} vs \{C\}
  - Etc.
- Contrast = linear combination of effects of alternatives
Important Points

- Use one-factor ANOVA to separate total variation into:
  - Variation within one system
    - Due to random errors
  - Variation between systems
    - Due to real differences (+ random error)

- Is the variation due to real differences statistically greater than the variation due to errors?
Generalized design of experiments

● Goals
  ● Isolate effects of each input variable.
  ● Determine effects of interactions.
  ● Determine magnitude of experimental error
  ● Obtain maximum information for given effort

● Basic idea
  ● Expand 1-factor ANOVA to \( m \) factors
Terminology

- **Response variable**
  - Measured output value: e.g., total execution time

- **Factors**
  - Input variables that can be changed
    - E.g.: cache size, clock rate, bytes transmitted

- **Levels**
  - Specific values of factors:
    - Continuous (~bytes) or discrete (type of system)

- **Replication**
  - Completely re-run experiment with same input levels

- **Interaction**
  - Effect of input factor A depends on level of input factor B
Two-factor experiments

- Two factors (inputs)
  - A, B
- Separate total variation in output values into:
  - Effect due to A
  - Effect due to B
  - Effect due to interaction of A and B (AB)
  - Experimental error
Example – User response time

- A = degree of multiprogramming
- B = memory size
- AB = interaction of memory size and degree of multiprogramming

<table>
<thead>
<tr>
<th>A</th>
<th>B (Mbytes)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>32</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>0.52</td>
</tr>
<tr>
<td>3</td>
<td>0.81</td>
</tr>
<tr>
<td>4</td>
<td>1.50</td>
</tr>
</tbody>
</table>
Two-factor ANOVA

- Factor A – $a$ input levels
- Factor B – $b$ input levels
- $n$ measurements for each input combination
- $abn$ total measurements
Two-factor ANOVA

- Each individual measurement is composition of
  - Overall mean
  - Effects
  - Interactions
  - Measurement errors
Example

- Output = user response time (seconds)
- Want to separate effects due to
  - A = degree of multiprogramming
  - B = memory size
  - AB = interaction
  - Error
- Need replications to separate error

<table>
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</tbody>
</table>
Conclusions from example

- 77.6% (SSA/SST) of all variation in response time due to degree of multiprogramming
- 11.8% (SSB/SST) due to memory size
- 9.9% (SSAB/SST) due to interaction
- 0.7% due to measurement error
- 95% confident that all effects and interactions are statistically significant
A problem

- Full factorial design with replication
  - Measure system response with all possible input combinations
  - Replicate each measurement $n$ times to determine effect of measurement error
- $m$ factors, $v$ levels, $n$ replications
  \[ \rightarrow n \ v^m \text{ experiments} \]
- $m = 5$ input factors, $v = 4$ levels, $n = 3$
  \[ \rightarrow 3(4^5) = 3,072 \text{ experiments!} \]
Fractional factorial designs: $n2^m$ experiments

- Special case of generalized $m$-factor experiments
- Restrict each factor to two possible values
  - High, low
  - On, off
- Find factors that have largest impact
- Full factorial design with only those factors
Still too many experiments with $n2^m$!

- Plackett and Burman designs (1946)
  - Multifactorial designs
- Effects of main factors only
  - Logically minimal number of experiments to estimate effects of $m$ input parameters (factors)
  - Ignores interactions
- Requires $O(m)$ experiments
  - Instead of $O(2^m)$ or $O(v^m)$