Network Optimization by Randomization
(Distributed Computing):

Maximal Independent Set
What is a MIS?

MIS

An independent set (IS) of an undirected graph is a subset \( U \) of nodes such that no two nodes in \( U \) are adjacent. An IS is maximal if no node can be added to \( U \) without violating IS (called MIS). A maximum IS (called MaxIS) is one of maximum cardinality.

... known from „classic TCS“: applications? backbone, parallelism, ... complexities?
MIS and MaxIS?
Nothing, IS, MIS, MaxIS?

IS but not MIS.
Nothing, IS, MIS, MaxIS?

Nothing.
Nothing, IS, MIS, MaxIS?

MIS.
Nothing, IS, MIS, MaxIS?

MaxIS.
Complexities?

MaxIS is NP-hard!
So let’s concentrate on MIS...

How much worse can MIS be than MaxIS?
MIS vs MaxIS

How much worse can MIS be than MaxIS?

minimal MIS? maxIS?
MIS vs MaxIS

How much worse can MIS be than Max-IS?

minimal MIS?  

maxIS?
How to compute a MIS in a distributed manner?!
Recall: Local Algorithm

Send...

... receive...

... compute.
Slow MIS

assume node IDs
Each node $v$:
1. If all neighbors with larger IDs have decided not to join MIS then:
   $v$ decides to join MIS

Analysis?
Analysis

**Time Complexity?**

Not faster than sequential algorithm!
Worst-case example?
E.g., sorted line: \( O(n) \) time.

**Local Computations?**

Fast! 😊

**Message Complexity?**

For example in clique: \( O(n^2) \)
\((O(m) \text{ in general: each node needs to inform all neighbors when deciding.})\)
MIS and Colorings

Independent sets and colorings are related: how?

Each color in a valid coloring constitutes an independent set (but not necessarily a MIS).

How to compute MIS from coloring?

Choose all nodes of first color. Then for any additional color, add in parallel as many nodes as possible!

Why, and implications?
Coloring vs MIS

Valid coloring:
Coloring vs MIS

Independent set:
Coloring vs MIS

Add all possible blue:
Coloring vs MIS

Add all possible violet:
Coloring vs MIS

Add all possible green:
Coloring vs MIS

That's all: MIS!

Analysis of algorithm?
Analysis

Why does algorithm work?

Same color: all nodes independent, can add them in parallel without conflict (not adding two conflicting nodes concurrently).

Runtime?

Lemma

Given a coloring algorithm with runtime $T$ that needs $C$ colors, we can construct a MIS in time $C+T$. 
Discussion

What does it imply for MIS on trees?

We can color trees in log* time and with 3 colors, so:

**MIS on Trees**

There is a deterministic MIS on trees that runs in distributed time $O(\log^* n)$.
Better MIS Algorithms

Any ideas?

**Takeaway**
If you can’t find fast deterministic algorithms, try randomization!

Ideas for randomized algorithms?
Fast MIS from 1986...

**Fast MIS (1986)**

Proceed in rounds consisting of phases

In a **phase**:

1. each node $v$ **marks itself with probability $1/2d(v)$**
   where $d(v)$ denotes the current degree of $v$

2. if no **higher degree neighbor** is marked, $v$ joins MIS; otherwise, $v$ unmarks itself again (break ties arbitrarily)

3. **delete** all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

**Why is it correct? Why IS? Why MIS?**

Note: the higher the degree the less likely to mark, but the more likely to join MIS once marked!

Stefan Schmid @ T-Labs, 2011
MIS 1986

Probability of marking?

Fast MIS (1986)

Proceed in rounds consisting of phases

In a phase:

1. each node $v$ marks itself with probability $\frac{1}{2d(v)}$ where $d(v)$ denotes the current degree of $v$
2. if no higher degree neighbor is marked, $v$ joins MIS; otherwise, $v$ unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore
MIS 1986

Probability of marking?

Fast MIS (1986)

Proceed in rounds consisting of phases
In a phase:
1. each node $v$ marks itself with probability $1/2d(v)$
   where $d(v)$ denotes the current degree of $v$
2. if no higher degree neighbor is marked, $v$ joins MIS; otherwise, $v$ unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore
MIS 1986

Marking... Who stays?

Fast MIS (1986)

Proceed in rounds consisting of phases
In a phase:
1. each node \( v \) marks itself with probability \( 1/2d(v) \)
   where \( d(v) \) denotes the current degree of \( v \)
2. If no higher degree neighbor is marked, \( v \) joins MIS; otherwise, \( v \) unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore
MIS 1986

And now?

Fast MIS (1986)
Proceed in rounds consisting of phases
In a phase:
1. each node $v$ marks itself with probability $1/2d(v)$
   where $d(v)$ denotes the current degree of $v$
2. If no higher degree neighbor is marked, $v$ joins
   MIS; otherwise, $v$ unmarks itself again (break ties
   arbitrarily)
3. delete all nodes that joined the MIS plus their
   neighbors, as they cannot join the MIS anymore

unmarked: higher degree neighbor marked...
unmarked: tie broken...
MIS 1986

Delete neighborhoods...

Fast MIS (1986)
Proceed in rounds consisting of phases
In a phase:
1. each node $v$ marks itself with probability $1/2d(v)$
   where $d(v)$ denotes the current degree of $v$
2. if no higher degree neighbor is marked, $v$ joins MIS; otherwise, $v$ unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore
Correctness

**Fast MIS (1986)**

Proceed in rounds consisting of phases

In a phase:

1. each node \( v \) marks itself with probability \( 1/2d(v) \)
   where \( d(v) \) denotes the current degree of \( v \)
2. if no higher degree neighbor is marked, \( v \) joins MIS; otherwise, \( v \) unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

**IS**: Step 1 and Step 2 ensure that node only joins if neighbors do not!

**MIS**: At some time, nodes will mark themselves in Step 1.
Fast MIS (1986)

Proceed in rounds consisting of phases
In a phase:

1. each node v marks itself with probability $1/2d(v)$ where $d(v)$ denotes the current degree of v
2. if no higher degree neighbor is marked, v joins MIS; otherwise, v unmarks itself again (break ties arbitrarily)
3. delete all nodes that joined the MIS plus their neighbors, as they cannot join the MIS anymore

Runtime: how fast will algorithm terminate?
Our Strategy!

**We want to show logarithmic runtime. So for example?**

Idea:

Each node is removed with constant probability (e.g., $\frac{1}{2}$) in each round $\Rightarrow$ half of the nodes vanish in each round.

Or: Each edge is removed with constant probability in each round! As $O(\log m) = O(\log n^2) = O(\log n)$

Unfortunately, this is not true... 😞 Alternative?

A constant fraction of all nodes are removed in each step!
E.g., a constant subset of nodes is „good“ and a constant fraction thereof is removed...

Or the same for edges...
Analysis

Joining MIS

Node $v$ joins MIS in Step 2 with probability $p \geq ?$

Proof.

On what could it depend?

Marked with probability that depends on degree, i.e., $1/2d(v)$. (So at most this...)

In MIS subsequently if degree is largest... (This is likely then if degree is small!)

We will find that marked nodes are likely to join MIS!
Analysis

Joining MIS

Node $v$ joins MIS in Step 2 with probability $p \geq 1/(4d(v))$.

Proof.

Let $M$ be the set of marked nodes in Step 1.
Let $H(v)$ be the set of neighbors of $v$ with higher degree (or same degree and higher identifier).

$$P[v \notin MIS \mid v \in M] = P[\exists w \in H(v), w \in M \mid v \in M] = P[\exists w \in H(v), w \in M] \quad \text{// independent whether $v$ is marked or not}$$

$$\leq \sum_{w \in H(v)} P[w \in M] \quad \text{// do not only count exactly one but also multiple}$$

$$= \sum_{w \in H(v)} 1/(2d(w)) \quad \text{// see Joining MIS algorithm}$$

$$\leq \sum_{w \in H(v)} 1/(2d(v)) \quad \text{// $v$'s degree is the lowest one}$$

$$\leq d(v)/(2d(v)) = 1/2 \quad \text{// at most $d(v)$ higher neighbors...}$$

So

$$P[v \in MIS] = P[v \in MIS \mid v \in M] \cdot P[v \in M] \geq \frac{1}{2} \cdot 1/(2d(v))$$

Marked nodes are likely to be in MIS!

QED

Stefan Schmid @ T-Labs, 2011
Recall Our Strategy!

We want to show logarithmic runtime. So for example?

Idea:

Each node is removed with constant probability (e.g., \(\frac{1}{2}\)) in each round => half of the nodes vanish in each round.

Or: Each edge is removed with constant probability in each round! As \(O(\log m) = O(\log n^2) = O(\log n)\)

Unfortunately, this is not true... 😞 Alternative?

Let's try this:

A constant fraction of all nodes are removed in each step!
E.g., a constant subset of nodes is „good“ and a constant fraction thereof is removed...

How to define good nodes?!
Node with low degree neighbors!
(Why? Likely to be removed as neighbors are likely to be marked and hence join MIS...)
Analysis

**Good&Bad Nodes**

A node $v$ is called *good* if

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \frac{1}{6}. $$

What does it mean?
A good node has neighbors of low degree. Likely to be removed when neighbor joins MIS!

**Good Nodes**

A good node $v$ will be removed in Step 3 with probability

$$p \geq \frac{1}{36}. $$

Proof?
Proof („Good Nodes“).

Goal: A good node $v$ will be removed in Step 3 with probability $p \geq 1/36$.

If $v$ has a neighbor $w$ with $d(w) \leq 2$?

Done: „Joining MIS“ lemma implies that prob. to remove at least $1/8$ since neighbor $w$ will join...

So let’s focus on neighbors with degree at least 3: thus for any neighbor $w$ of $v$ we have $1/(2d(w)) \leq 1/6$. 
Analysis (2)

Proof ("Good Nodes").

Goal:

- **Good Nodes**
  A good node $v$ will be removed in Step 3 with probability $p \geq 1/36$.

"Assets":

- **Joining MIS**
  Node $v$ joins MIS in Step 2 with probability $p \geq 1/(4d(v))$.

- **Good&Bad Nodes**
  A node $v$ is called good if $\sum_{w \in N(v)} 1/(2d(w)) \geq 1/6$.

So neighbors have degree at least 3...

Then, for a good node $v$, there must be a subset $S \subseteq N(v)$ such that

$$
\frac{1}{6} \leq \sum_{w \in S} \frac{1}{2d(w)} \leq \frac{1}{3}.
$$

Why?

By taking all neighbors we have at least $1/6$ (Definition), and we can remove individual nodes with a granularity of at least $1/6$ (degree at least 3).
Analysis (3)

Proof („Good Nodes“).

Let $R$ be event that $v$ is removed (e.g., if neighbor joins MIS).

$$P[R] \geq P[\exists u \in S, u \in MIS] \quad \text{// removed e.g., if neighbor joins}$$

$$\geq \sum_{u \in S} P[u \in MIS] - \sum_{u,w \in S} P[u \in MIS \text{ and } w \in MIS] \quad \text{// why?}$$

By truncating the inclusion-exclusion principle...:
Probability that there is one is sum of probability for all individual minus probability that two enter, plus...

$$P[R] \geq \sum_{u \in S} P[u \in MIS] - \sum_{u,w \in S} P[u \in MIS \text{ and } w \in MIS]$$

just derived!

$$\geq \sum_{u \in S} \frac{1}{4d(u)} - \sum_{u \in S} \sum_{w \in S} \frac{1}{2d(u)} \cdot \frac{1}{2d(w)}$$

independent but count same node double in sum...

see Joining MIS lemma

see Joining MIS lemma

$$\geq \sum_{u \in S} \frac{1}{2d(u)} \left( \frac{1}{2} - \sum_{w \in S} \frac{1}{2d(w)} \right) \geq \frac{1}{6} \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{36}.$$
Analysis

We just proved:

A good node $v$ will be removed in Step 3 with probability

$$p \geq 1/36.$$ 

Cool, good nodes have Constant probability! 😊

But what now?

What does it help?

Are many nodes good in a graph?

Example: in star graph, only single node is good... 😞

But: there are many „good edges“...

How to define good edges?

Idea: edge is removed if either of its endpoints are removed! So good if at least one endpoint Is a good node! And there are many such edges...
Analysis

**Good & Bad Edges**

An edge $e=(u,v)$ called *bad* if both $u$ and $v$ are bad (not good). Else the edge is called good.

A bad edge is incident to two nodes with neighbors of high degrees.

**Good Edges**

At least half of all edges are good, at any time.

Proof?
Analysis

Not many good nodes...

... but many good edges!
Analysis

Idea: Construct an auxiliary graph! Direct each edge towards higher degree node (if both nodes have same degree, point it to one with higher ID).

**Helper Lemma**

A bad node $v$ has out-degree at least twice its indegree.

**Proof („Helper Lemma“).**

Assume the opposite: at least one third of the neighbor nodes (let’s call them $S$) have degree at most $d(v)$. But then

$$\sum_{w \in N(v)} \frac{1}{2d(w)} \geq \sum_{w \in S} \frac{1}{2d(w)} \geq \sum_{w \in S} \frac{1}{2d(v)} \geq \frac{d(v)}{3} \cdot \frac{1}{2d(v)} = \frac{1}{6}$$

**Contradiction:** $v$ would be good!

QED
Analysis

Idea: Construct an auxiliary graph! Direct each edge towards higher degree node (if both nodes have same degree, point it to one with higher ID).

Helper Lemma

A bad node $v$ has out-degree at least twice its indegree.

So what?

The number of edges into bad nodes can be at most half the number of all edges!
So at least half of all edges are directed into good nodes!
And they are good! 😊
Analysis

**Fast MIS (1986)**

Fast MIS terminates in expected time $O(\log n)$.

**Proof („Fast MIS“)?**

We know that a good node will be deleted with constant probability in Step 3. And with it, a good edge (by definition)! Since at least half of all the edges are good (and thus have at least one good incident node), a constant fraction of edges will be deleted in each phase. (Note that $O(\log m)=O(\log n)$.)

QED
Back to the future:
Fast MIS from 2009...!

Even simpler algorithm!
Fast MIS from 2009...

Fast MIS (2009)

Proceed in rounds consisting of phases
In a phase:

1. each node chooses a random value $r(v) \in [0,1]$ and sends it to its neighbors.
2. If $r(v) < r(w)$ for all neighbors $w \in N(v)$, node $v$ enters the MIS and informs the neighbors.
3. If $v$ or a neighbor of $v$ entered the MIS, $v$ terminates (and $v$ and edges are removed), otherwise $v$ enters next phase!
Fast MIS from 2009...
Fast MIS from 2009...

Choose random values!
Fast MIS from 2009...

Min in neighborhood => IS!
Fast MIS from 2009...

Remove neighborhoods...
Fast MIS from 2009...

Choose random values!
Fast MIS from 2009...

Min in neighborhood => IS!
Fast MIS from 2009...

Remove neighborhoods...
Fast MIS from 2009...

Choose random values!
Fast MIS from 2009...

lowest value => IS
Fast MIS from 2009...

... done: MIS!
Fast MIS (2009)

Proceed in rounds consisting of phases
In a phase:
1. each node chooses a random value \( r(v) \in [0,1] \) and sends it to its neighbors.
2. If \( r(v) < r(w) \) for all neighbors \( w \in N(v) \), node \( v \) enters the MIS and informs the neighbors.
3. If \( v \) or a neighbor of \( v \) entered the MIS, \( v \) terminates (and \( v \) and edges are removed), otherwise \( v \) enters next phase!

Why is it correct? Why IS?
Step 2: if \( v \) joins, neighbors do not
Step 3: if \( v \) joins, neighbors will never join again
Fast MIS from 2009...

Fast MIS (2009)

Proceed in rounds consisting of phases
In a phase:
1. each node chooses a random value \( r(v) \in [0,1] \) and sends it to its neighbors.
2. If \( r(v) < r(w) \) for all neighbors \( w \in N(v) \), node \( v \) enters the MIS and informs the neighbors.
3. If \( v \) or a neighbor of \( v \) entered the MIS, \( v \) terminates (and \( v \) and edges are removed), otherwise \( v \) enters next phase!

Why MIS?

Node with largest random value will always join the IS, so there is always progress.
Fast MIS (2009)

Proceed in rounds consisting of phases
In a phase:

1. each node chooses a random value \( r(v) \in [0,1] \) and sends it to its neighbors.
2. If \( r(v) < r(w) \) for all neighbors \( w \in N(v) \), node \( v \) enters the MIS and informs the neighbors.
3. If \( v \) or a neighbor of \( v \) entered the MIS, \( v \) terminates (and \( v \) and edges are removed), otherwise \( v \) enters next phase!

Runtime?
Analysis: Recall „Linearity of Expectation“

Theorem 5.9 (Linearity of Expectation). Let $X_i$, $i = 1, \ldots, k$ denote random variables, then

$$\mathbb{E} \left[ \sum_i X_i \right] = \sum_i \mathbb{E} [X_i].$$

Proof. It is sufficient to prove $\mathbb{E} [X + Y] = \mathbb{E} [X] + \mathbb{E} [Y]$ for two random variables $X$ and $Y$, because then the statement follows by induction. Since

$$P [(X,Y) = (x,y)] = P [X = x] \cdot P [Y = y | X = x]$$
$$= P [Y = y] \cdot P [X = x | Y = y]$$

we get that

$$\mathbb{E} [X + Y] = \sum_{(X,Y) = (x,y)} P [(X,Y) = (x,y)] \cdot (x + y)$$
$$= \sum_{X=x} \sum_{Y=y} P [X = x] \cdot P [Y = y | X = x] \cdot x$$
$$+ \sum_{Y=y} \sum_{X=x} P [Y = y] \cdot P [X = x | Y = y] \cdot y$$
$$= \sum_{X=x} P [X = x] \cdot x + \sum_{Y=y} P [Y = y] \cdot y$$
$$= \mathbb{E} [X] + \mathbb{E} [Y].$$

We sum over all possible $y$ values for a given $x$, so $=1$
We want to show that also this algorithm has logarithmic runtime! How?

Idea: if per phase a constant fraction of node disappeared, it would hold! (Recall definition of logarithm...)

Again: this is not true unfortunately... 😞

Alternative proof? Similar to last time?

Show that any edge disappears with constant probability!

But also this does not work: edge does not have constant probability to be removed!

But maybe edges still vanish quickly...?

Let’s estimate the number of disappearing edges per round again!
Analysis? (2)

Probability of a node $v$ to enter MIS?

Probability = node $v$ has largest ID in neighborhood, so at least $1/(d(v)+1)$...

... also $v$‘s neighbors‘ edges will disappear with this probability, so more than $d(v)$ edges go away with this probability!

But let’s make sure we do not double count edges!

Idea: only count edges from a neighbor $w$ when $v$ is the smallest value even in $w$‘s neighborhood! It’s a subset only, but sufficient!
Edge Removal: Analysis (1)

Edge Removal
In expectation, we remove at least half of all the edges in any phase.

Proof („Edge Removal“)?
Consider the graph \( G=(V,E) \), and assume \( v \) joins MIS (i.e., \( r(v)<r(w) \) for all neighbors \( w \)).
If in addition, it holds that \( r(v)<r(x) \) for all neighbors \( x \) of a neighbor \( w \), we call this event \( (v \Rightarrow w) \).

What is the probability of this event (that \( v \) is minimum also for neighbors of the given neighbor)?

\[
P[(v \Rightarrow w)] \geq \frac{1}{d(v)+d(w)},
\]

since \( d(v)+d(w) \) is the maximum possible number of nodes adjacent to \( v \) and \( w \).
If \( v \) joins MIS, all edges \((w,x)\) will be removed; there are at least \( d(w) \) many.
Edge Removal

In expectation, we remove at least half of all the edges in any phase.

Proof ("Edge Removal")?

How many edges are removed?
Let $X_{(v \rightarrow w)}$ denote random variable for number of edges adjacent to $w$ removed due to event $(v \rightarrow w)$. If $(v \rightarrow w)$ occurs, $X_{(v \rightarrow w)}$ has value $d(w)$, otherwise 0.
Let $X$ denote the sum of all these random variables.

So:

$$
\mathbb{E}[X] = \sum_{\{v,w\} \in E} \mathbb{E}[X_{(v \rightarrow w)}] + \mathbb{E}[X_{(w \rightarrow v)}]
$$

$$
= \sum_{\{v,w\} \in E} P[\text{Event } (v \rightarrow w)] \cdot d(w) + P[\text{Event } (w \rightarrow v)] \cdot d(v)
$$

$$
\geq \sum_{\{v,w\} \in E} \frac{d(w)}{d(v) + d(w)} + \frac{d(v)}{d(w) + d(v)}
$$

$$
= \sum_{\{v,w\} \in E} 1 = |E|.
$$

So all edges gone in one phase?! We still overcount!
Edge Removal

In expectation, we remove at least half of all the edges in any phase.

**Proof ("Edge Removal")?**

We still overcount:
Edge \{v,w\} may be counted twice: for event (u=>v) and event (x=>w).

However, it cannot be more than twice, as there is at most one event (*=v) and at most one event (=*w):

Event (u=>v) means r(u)<r(w) for all w ∈ N(v); another (u'='v) would imply that r(u')>r(u) ∈ N(v).

**So at least half of all edges vanish!**

QED
MIS of 2009

Expected running time is $O(\log n)$.

Proof („MIS 2009“)?

Number of edges is cut in two in each round...

QED

Actually, the claim even holds with high probability! (see „Skript“)
Excursion: Matchings

Matching

A matching is a subset $M$ of edges $E$ such that no two edges in $M$ are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.
Excursion: Matchings

Maximal.
Excursion: Matchings

Excursion: Matchings

Maximum but not perfect.
Discussion: Matching

Matching

A matching is a subset $M$ of edges $E$ such that no two edges in $M$ are adjacent.
A maximal matching cannot be augmented.
A maximum matching is the best possible.
A perfect matching includes all nodes.

How to compute with an IS algorithm?
Discussion: Matching

An IS algorithm is a matching algorithm! How?

For each edge in original graph make vertex, connect vertices if their edges are adjacent.
Discussion: Matching

MIS = maximal matching: matching does not have adjacent edges!
How to use a MIS algorithm for graph coloring?

Clone each node $v$, $d(v)+1$ many times. Connect clones completely and edges from $i$-th clone to $i$-th clone. Then?
Run MIS: if $i$-th copy is in MIS, node gets color $i$. 

Discussion: Graph Coloring
Discussion: Graph Coloring

Example:

How to use a MIS algorithm for graph coloring?

MIS Coloring

a => blue
b => green
Discussion: Graph Coloring

Why does it work?

1. Idea conflict-free: adjacent nodes cannot get same color (different index in MIS, otherwise adjacent!), and each node has at most one clone in IS, so valid.

2. Idea colored: each node gets color, i.e., each node has a clone in IS: there are only $d(v)$ neighbor clusters, but our cluster has $d(v)+1$ nodes...
Discussion: Dominating Set

**Dominating Set**

A subset $D$ of nodes such that each node either is in the dominating set itself, or one of its neighbors is (or both).

How to compute a dominating set?
See Skript. 😊
Literature for further reading:

- Peleg‘s book (as always 😊 )